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Twisted spin Sutherland models from quantum Hamiltonian reduction

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Abstract
Recent general results on Hamiltonian reductions under polar group actions are applied to study some reductions of the free particle governed by the Laplace–Beltrami operator of a compact, connected, simple Lie group. The reduced systems associated with arbitrary finite-dimensional irreducible representations of the group by using the symmetry induced by twisted conjugations are described in detail. These systems generically yield integrable Sutherland-type many-body models with spin, which are called twisted spin Sutherland models if the underlying twisted conjugations are built on non-trivial Dynkin diagram automorphisms. The spectra of these models can be calculated, in principle, by solving certain Clebsch–Gordan problems, and the result is presented for the models associated with the symmetric tensorial powers of the defining representation of $SU(N)$.

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1. Introduction

The investigation of one-dimensional integrable many-body systems initiated by Calogero [1], Sutherland [2] and others is still an actively pursued field of mathematical physics. These models possess interesting physical applications and are closely related to harmonic analysis and to the theory of special functions. See, e.g., the reviews in [3–10].

The aim of this paper is to apply the quantum Hamiltonian reduction approach developed in [11, 12] under certain general assumptions to construct and analyze new examples of spin

Sutherland-type models, where by ‘Sutherland type’ we mean that the interaction potential involves the function $1/\sin^2 x$ in association with a root system. A first impression about our models may be obtained by viewing the special cases in equations (5.19) and (5.20), where the operators in the numerators act on internal ‘spin’ degrees of freedom and the presence of $1/\cos^2 x$ in the interaction indicates the twisted character of these Sutherland-type models. These examples illustrate the statement [11] that quantum Hamiltonian reductions of the free particle on a Lie group or on a symmetric space under polar actions of compact symmetry groups lead to Calogero–Sutherland-type models with internal degrees of freedom in general.

A polar action of a compact Lie group, G, is an isometric action on a complete Riemannian manifold, Y, which permits the introduction of adapted polar coordinates as defined in [13] systematizing classical examples. In the cases of our interest the radial coordinates run over a suitable Abelian Lie group, whose Lie algebra carries an associated root system. The quantum Hamiltonian reduction amounts to restricting the Laplace–Beltrami operator, Δ_Y, of Y to vector-valued generalized spherical functions, which are wavefunctions on Y belonging to some fixed representation type under the symmetry group G. If this representation is the trivial one, then the reduction yields the radial part of the Laplace–Beltrami operator.

It was a pioneering observation of Olshanetsky and Perelomov [14] that the radial part of the Laplace–Beltrami operator of any Riemannian symmetric space provides the Hamiltonian of a Calogero–Sutherland-type model at special coupling constants. One may also note by inspecting examples that if one considers spherical functions corresponding to an arbitrary representation of the symmetry group, then the angular part of the Laplace–Beltrami operator contributes an interaction term of spin Calogero–Sutherland type for general families of polar actions (e.g., for the so-called Hermann actions recalled in section 3). We say that ‘spin’ degrees of freedom are present to express the fact that the reduced wavefunctions are vector valued in all but some exceptional cases. It is also important to note that the reduced wavefunctions are actually scalar valued for certain non-trivial representation types under some symmetry groups. This was pointed out by Etingof, Frenkel and Kirillov in [15], where they used this observation to derive the standard Sutherland model with arbitrary integer coupling constants from Hamiltonian reduction of the free particle on the group manifold $SU(N)$.

In standard harmonic analysis [16], the spherical functions and the spectrum of the Laplace–Beltrami operator are among the central objects of interest. As discussed above, it is well known that several powerful results of harmonic analysis can be translated into statements about many-body models with spin defined by Hamiltonian reduction. However, it appears that this idea has not yet been systematically exploited. In fact, since the standard harmonic analysis approach has the limitation of giving spinless models only at special coupling constants, the attention was mainly focused on the more algebraic methods that are capable to overcome this limitation, such as the techniques relying on Dunkl operators and Hecke algebras (reviewed in [4, 5, 7, 10]). Our opinion is that although the standard harmonic analysis approach gives indeed only a limited class of spinless many-body models, it would be worth to develop this point of view systematically, partly since the resulting models with spin are interesting and partly since it is still not clear what is the full set of spinless models that can be described in this framework. For recent studies concerning the spinless models, see [17, 18].

The program to develop the classical and the quantum Hamiltonian reduction approach to spin Calogero–Sutherland-type models in general terms and to explore the set of systems that it covers was advanced in the recent papers [11, 19–21]. In this paper we deal with certain novel examples at the quantum-mechanical level, which we call twisted spin Sutherland models since they result from quantum Hamiltonian reduction based on the so-called twisted conjugation action of a compact simple Lie group on itself. The definition of this polar action can be seen in equation (3.12), where Θ is an automorphism of the compact symmetry group G. It generalizes the ordinary conjugation action, which is recovered if Θ is the identity automorphism. The geometry of twisted conjugations has been recently investigated in [22–24], and we shall use some results of these references. Our work builds also on [20], where we have described the classical mechanical counterparts of the twisted spin Sutherland models.

The content of the present paper and our main results can be outlined as follows. In section 2, we review the quantum Hamiltonian reduction of the free particle focusing mainly on the description of the reduced systems obtained with the aid of polar actions of compact Lie groups. This section is based on our earlier work [11, 12] (see also [25]). Section 3 contains the derivation of the spin Sutherland models associated with involutive Dynkin diagram automorphisms of the simple Lie algebras. The models are displayed in proposition 3.1, which is a new result. The twisted spin Sutherland models correspond to non-trivial automorphisms, but the previously studied case (e.g. [15]) of the trivial automorphism is also covered. Section 4 is devoted to explaining how the diagonalization of the spin Sutherland Hamiltonians of proposition 3.1 can be performed in principle by solving certain Clebsch–Gordan problems in the representation theory of the underlying symmetry group G. The construction of the models involves choosing a representation of G, and in section 5 we analyze the examples associated with the symmetric tensorial powers of the defining representation of $G = SU(N)$. We first present these models, in propositions 5.1 and 5.2, by using the realization of the symmetric tensors in terms of N harmonic oscillators. We then determine the spectrum of the Hamiltonian in these cases by applying some standard results (the so-called Pieri formulae) of representation theory. The spectra of these twisted spin Sutherland Hamiltonians are given by theorem 5.3, which is one of our main results. This generalizes the well-known formula (5.33) of the spectrum of the standard spinless Sutherland model (5.30), here recovered from Hamiltonian reduction, at integer couplings, as a warm-up exercise [15]. Finally, our conclusions and comments on open problems are collected in section 6 and the three appendices contain some technical details.

In what follows we make an effort to present the analysis in a self-contained manner, as an application of a general framework that can be used in future works as well.

2. Some facts about quantum Hamiltonian reduction

We here summarize basic facts about quantum Hamiltonian reductions of free particles on complete Riemannian manifolds under isometric actions of compact Lie groups. The free Hamiltonian will be taken to be the scalar Laplace–Beltrami operator, and we shall assume the existence of generalized polar coordinates adapted to the group action. No new results are contained in this section, which mainly serves to fix notations for the subsequent developments. Proofs and more details can be found in [11, 12, 25].

2.1. Definitions

Suppose that a compact Lie group G acts on a complete, connected, smooth Riemannian manifold (Y, η) in an isometric manner. This means that we are given a smooth left-action

$$\phi : G \times Y \to Y, \qquad (g, y) \mapsto \phi(g, y) = \phi_g(y) = g \cdot y \tag{2.1}$$

of G on Y satisfying $\phi_g^* \eta = \eta$ for every $g \in G$. Then the measure μ_Y on Y induced by the metric η is G-invariant, and the natural action of G on the Hilbert space $L^2(Y, \mathrm{d}\mu_Y)$ gives rise to a continuous unitary representation of G,

$$U : G \to \mathcal{U}(L^2(Y, \mathrm{d}\mu_Y)), \qquad g \mapsto U(g). \tag{2.2}$$

Obviously, this unitary representation of G commutes with the restriction, Δ_Y^0, of the Laplace–Beltrami operator, Δ_Y, to the space of smooth complex functions of compact support, $C_c^\infty(Y)$,

$$\Delta_Y^0 U(g) f = U(g) \Delta_Y^0 f \quad \forall g \in G, \qquad \forall f \in C_c^\infty(Y). \tag{2.3}$$

The Laplace–Beltrami operator with domain $C_c^\infty(Y)$,

$$\Delta_Y^0 := \Delta_Y\big|_{C_c^\infty(Y)} : C_c^\infty(Y) \to C_c^\infty(Y), \tag{2.4}$$

is an essentially self-adjoint operator on $L^2(Y, \mathrm{d}\mu_Y)$, i.e., its closure $\bar{\Delta}_Y^0$ is self-adjoint (see, e.g., [26]). As a result, the pair $\left(L^2(Y, \mathrm{d}\mu_Y), -\frac{1}{2}\bar{\Delta}_Y^0\right)$ is a quantum-mechanical system with symmetry group G, which is a quantum-mechanical analog of the free classical point mass moving along geodesics on the Riemannian manifold (Y, η).

Let $\rho : G \to \mathcal{U}(V_\rho)$ be a continuous unitary irreducible representation (finite-dimensional 'irrep') of G, and denote the corresponding complex conjugate representation by (ρ^*, V_{ρ^*}). It is not difficult to exhibit the following unitary equivalence of G-representations:

$$L^2(Y, \mathrm{d}\mu_Y) \cong \oplus_\rho L^2(Y, V_\rho, \mathrm{d}\mu_Y)^G \otimes V_{\rho^*} \tag{2.5}$$

where the sum is over the pairwise inequivalent irreps and $L^2(Y, V_\rho, \mathrm{d}\mu_Y)^G$ is the space of G-singlets in the Hilbert space of V_ρ-valued square-integrable functions on Y. (Of course, $L^2(Y, V_\rho, \mathrm{d}\mu_Y) \cong L^2(Y, \mathrm{d}\mu_Y) \otimes V_\rho$ as a representation space of G, and its scalar product also uses the scalar product on V_ρ.) In terms of the decomposition (2.5) of $L^2(Y, \mathrm{d}\mu_Y)$, the action of G is non-trivial only on the factors V_{ρ^*}, and the action of $\bar{\Delta}_Y^0$ is non-trivial only on the 'multiplicity spaces' $L^2(Y, V_\rho, \mathrm{d}\mu_Y)^G$. By keeping only these multiplicity spaces, one obtains a reduced quantum system for every G irrep (ρ, V_ρ).

To be more explicit, the reduced Hilbert space consists of the V_ρ-valued G-equivariant square-integrable functions on Y,

$$L^2(Y, V_\rho, \mathrm{d}\mu_Y)^G := \{f \mid f \in L^2(Y, V_\rho, \mathrm{d}\mu_Y),\ f \circ \phi_g = \rho(g) \circ f \forall g \in G\}. \tag{2.6}$$

The Laplace–Beltrami operator acts naturally on the space of the V_ρ-valued G-equivariant smooth functions of compact support, simply componentwise. This gives the operator

$$\Delta_\rho : C_c^\infty(Y, V_\rho)^G \to C_c^\infty(Y, V_\rho)^G, \tag{2.7}$$

which is essentially self-adjoint on the Hilbert space $L^2(Y, V_\rho, \mathrm{d}\mu_Y)^G$. The closure of Δ_ρ is the Hamiltonian of the reduced quantum system,

$$\left(L^2(Y, V_\rho, \mathrm{d}\mu_Y)^G, -\tfrac{1}{2}\bar{\Delta}_\rho\right), \tag{2.8}$$

that arises from the free particle in association with the irrep (ρ, V_ρ).

Although the reduced quantum system is already completely fixed by (2.8), it is desirable (e.g., for the physical interpretation) to realize the reduced state-space $L^2(Y, V_\rho, \mathrm{d}\mu_Y)^G$ as a Hilbert space of appropriate functions on the reduced configuration space $Y_{\mathrm{red}} := Y/G$ and the reduced Hamiltonian operator as a differential operator over this space. Here one encounters a difficulty since the orbit space Y/G is not a smooth manifold but a *stratified space* in general, i.e., a disjoint union of countably many smooth Riemannian manifolds. However, restricting to the generic points forming the submanifold $\check{Y} \subset Y$ of *principal orbit type*, one obtains a smooth fiber bundle $\pi : \check{Y} \to \check{Y}/G$, and the smooth part of the reduced configuration space, $\check{Y}_{\mathrm{red}} := \check{Y}/G$, can be endowed with a reduced Riemannian metric, η_{red}, in such a way that π becomes a Riemannian submersion[5]. The crucial facts are that $\check{Y}$ is dense and open in Y and its complement is of zero measure. Since the wavefunctions belonging to the domain of essential self-adjointness $C_c^\infty(Y, V_\rho)^G$ (2.7) can be recovered from their restriction to $\check{Y}$, the above-mentioned difficulty is only apparent. It is possible to work out a complete characterization

[5] For detailed discussions on the principal orbit type and stratifications, we recommend [27, 28].

of the reduced systems in terms of the smooth reduced configuration manifold $(\check{Y}_{\mathrm{red}}, \eta_{\mathrm{red}})$ in general. Next we present this under a simplifying assumption that holds in the examples of our interest.

2.2. *Characterization of the reduced systems*

From now on we assume that G acts on (Y, η) in a *polar* manner, that is, the action ϕ (2.1) admits *sections* in the sense of Palais and Terng [13]. Recall that a section $\Sigma \subset Y$ is a connected, closed, regularly embedded smooth submanifold of Y that meets every G-orbit and it does so orthogonally at every intersection point of Σ with an orbit. By its embedding, Σ inherits a Riemannian metric η_Σ, which induces a measure μ_Σ on Σ. For a section Σ, denote by $\check{\Sigma}$ a connected component of the manifold $\hat{\Sigma} := \check{Y} \cap \Sigma$. The isotropy subgroups of all elements of $\hat{\Sigma}$ are the same and for a fixed section we define $K := G_y$ for $y \in \hat{\Sigma}$. The restriction of $\pi: \check{Y} \to \check{Y}/G$ onto $\check{\Sigma}$ provides an isometric diffeomorphism between the Riemannian manifolds $(\check{\Sigma}, \eta_{\check{\Sigma}})$ and $(\check{Y}_{\mathrm{red}}, \eta_{\mathrm{red}})$, and the G-equivariant diffeomorphism

$$\check{\Sigma} \times (G/K) \ni (q, gK) \mapsto \phi_g(q) \in \check{Y} \tag{2.9}$$

defines a global trivialization of the fiber bundle $\pi: \check{Y} \to \check{Y}/G$. We below recall the characterization of the reduced systems in terms of the reduced configuration space $(\check{\Sigma}, \eta_{\check{\Sigma}}) \cong (\check{Y}_{\mathrm{red}}, \eta_{\mathrm{red}})$.

First, we introduce the $\mathbb{C}$-linear space

$$\mathrm{Fun}\big(\check{\Sigma}, V_\rho^K\big) := \big\{ f \in C^\infty\big(\check{\Sigma}, V_\rho^K\big) \big| \exists \mathcal{F} \in C_c^\infty(Y, V_\rho)^G,\ f = \mathcal{F}|_{\check{\Sigma}} \big\}, \tag{2.10}$$

where V_ρ^K is the subspace of *K-invariant vectors* in the representation space V_ρ. We assume that $\dim\big(V_\rho^K\big) > 0$. As a vector space, $\mathrm{Fun}\big(\check{\Sigma}, V_\rho^K\big)$ is naturally isomorphic to $C_c^\infty(Y, V_\rho)^G$ and can be equipped with a scalar product induced by this isomorphism. Using also that $C_c^\infty(Y, V_\rho)^G$ is dense in $L^2(Y, V_\rho, \mathrm{d}\mu_Y)^G$, for the closure of $\mathrm{Fun}\big(\check{\Sigma}, V_\rho^K\big)$ we obtain the Hilbert space isomorphism $\overline{\mathrm{Fun}}\big(\check{\Sigma}, V_\rho^K\big) \cong L^2(Y, V_\rho, \mathrm{d}\mu_Y)^G$.

We also introduce the density function, $\delta: \check{\Sigma} \to (0, \infty)$, as follows. The G-orbit $G \cdot q \subset Y$ through any point $q \in \check{\Sigma}$ is an embedded submanifold of Y and by its embedding it inherits a Riemannian metric, $\eta_{G\cdot g}$. We let

$$\delta(q) := \text{volume of the Riemannian manifold } (G \cdot q, \eta_{G\cdot q}), \tag{2.11}$$

where, of course, the volume is understood with respect to the measure belonging to $\eta_{G\cdot q}$.

Now let us consider the Lie algebra $\mathcal{G} := \mathrm{Lie}(G)$ and its subalgebra $\mathcal{K} := \mathrm{Lie}(K)$. Choose a G-invariant positive definite scalar product, $\mathcal{B}$, on $\mathcal{G}$, which gives rise to the orthogonal decomposition

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{K}^\perp. \tag{2.12}$$

For any $\xi \in \mathcal{G}$ denote by $\xi^\sharp$ the corresponding vector field on Y. At each point $q \in \check{\Sigma}$, the linear map

$$\mathcal{K}^\perp \ni \xi \mapsto \xi_q^\sharp \in T_q Y \tag{2.13}$$

is injective and permits to define the 'inertia operator' $\mathcal{J}(q) \in GL(\mathcal{K}^\perp)$ by requiring

$$\eta_q\big(\xi_q^\sharp, \zeta_q^\sharp\big) = \mathcal{B}(\xi, \mathcal{J}(q)\zeta) \quad \forall \xi, \zeta \in \mathcal{K}^\perp. \tag{2.14}$$

Note that $\mathcal{J}(q)$ is symmetric and positive definite with respect to the restriction of the scalar product $\mathcal{B}$ to $\mathcal{K}^\perp$. In $\mathcal{K}^\perp$ we introduce dual bases $\{T_\alpha\}$ and $\{T^\alpha\}$, $\mathcal{B}(T^\alpha, T_\beta) = \delta^\alpha_\beta$, and we let

$$b_{\alpha,\beta}(q) := \mathcal{B}(T_\alpha, \mathcal{J}(q)T_\beta), \qquad b^{\alpha,\beta}(q) := \mathcal{B}(T^\alpha, \mathcal{J}(q)^{-1}T^\beta). \tag{2.15}$$

The matrix $b^{\alpha,\beta}(q)$ is the inverse of $b_{\alpha,\beta}(q)$, and for the density function we have

$$\delta(q) = C\sqrt{|\det(b_{\alpha,\beta}(q))|} \quad \forall q \in \check{\Sigma}, \tag{2.16}$$

where $C > 0$ is some constant.

Finally, for the Lie algebra representation belonging to the unitary representation (ρ, V_ρ) of G, we introduce the notation $\rho' : \mathcal{G} \to u(V_\rho)$, where $u(V_\rho)$ is the Lie algebra of anti-hermitian operators on V_ρ. The following result is then proved in [11] (see also [12]).

Proposition 2.1. *Suppose that ϕ (2.1) is a polar action of the compact Lie group G on the Riemannian manifold (Y, η) and choose a section Σ for this action. Then, using the notations introduced above, the reduced system (2.8) associated with a continuous unitary irreducible representation (ρ, V_ρ) of G can be identified with the pair $\big(L^2\big(\check{\Sigma}, V_\rho^K, \mathrm{d}\mu_{\check{\Sigma}}\big), -\frac{1}{2}\bar{\Delta}_{\mathrm{red}}\big)$, where the reduced Laplace–Beltrami operator*

$$\Delta_{\mathrm{red}} := \Delta_{\check{\Sigma}} - \delta^{-\frac{1}{2}}\Delta_{\check{\Sigma}}(\delta^{\frac{1}{2}}) + b^{\alpha,\beta}\rho'(T_\alpha)\rho'(T_\beta) \tag{2.17}$$

with domain $\delta^{\frac{1}{2}}\mathrm{Fun}\big(\check{\Sigma}, V_\rho^K\big)$ is essentially self-adjoint on the Hilbert space $L^2\big(\check{\Sigma}, V_\rho^K, \mathrm{d}\mu_{\check{\Sigma}}\big)$, and $\bar{\Delta}_{\mathrm{red}}$ denotes its self-adjoint closure.

Remark 2.2. Proposition 2.1 utilizes the identification $L^2(Y, V_\rho, \mathrm{d}\mu_Y)^G \cong L^2\big(\check{\Sigma}, V_\rho^K, \delta\, \mathrm{d}\mu_{\check{\Sigma}}\big)$ (obtained by restricting the G-equivariant wavefunctions to $\check{\Sigma}$) as well as the isometry between $L^2\big(\check{\Sigma}, V_\rho^K, \delta\, \mathrm{d}\mu_{\check{\Sigma}}\big)$ and $L^2\big(\check{\Sigma}, V_\rho^K, \mathrm{d}\mu_{\check{\Sigma}}\big)$ defined by multiplying the restricted wavefunctions by $\delta^{\frac{1}{2}}$. The latter step is natural since the measure $\mu_{\check{\Sigma}}$ is directly defined by the reduced Riemannian metric $\eta_{\check{\Sigma}}$ that also enters $\Delta_{\check{\Sigma}}$.

Remark 2.3. One can view the reduced systems of proposition 2.1 from an alternative perspective that sheds light on a generalization of the well-known Weyl group invariance of the standard Calogero–Sutherland models. The essential point is that the restriction of the elements of $C_c^\infty(Y, V_\rho)^G$ to $\check{\Sigma}$ can be implemented in two steps, initially restricting them to $\hat{\Sigma} = \check{Y} \cap \Sigma$. The wave functions obtained in this first step are equivariant with respect to the residual symmetry transformations generated by the elements of G that map Σ (or equivalently $\hat{\Sigma}$) to itself. More precisely, since K acts trivially on Σ, these transformations form the factor group

$$W := N_G(\Sigma)/K, \tag{2.18}$$

where $N_G(\Sigma)$ contains the Σ-preserving elements of G. It is proved in [13] that W is a finite group for any polar action. The representation of G on V_ρ induces a representation of W on V_ρ^K, and W permutes the connected components of $\hat{\Sigma}$ by its action. We now have the following natural Hilbert space isomorphisms:

$$L^2(Y, V_\rho, \mathrm{d}\mu_Y)^G \cong L^2\big(\check{\Sigma}, V_\rho^K, \mathrm{d}\mu_{\check{\Sigma}}\big) \cong L^2\left(\hat{\Sigma}, V_\rho^K, \frac{1}{|W|}\mathrm{d}\mu_{\hat{\Sigma}}\right)^W \cong L^2\left(\Sigma, V_\rho^K, \frac{1}{|W|}\mathrm{d}\mu_{\Sigma}\right)^W, \tag{2.19}$$

where the last equality is based on the fact that $\Sigma \setminus \hat{\Sigma}$ has measure zero. The Hilbert space $L^2\big(\hat{\Sigma}, V_\rho^K, \frac{1}{|W|}\mathrm{d}\mu_{\hat{\Sigma}}\big)^W$ carries the Hamiltonian given by the same formula as Δ_{red} (2.17) but using $\hat{\Sigma}$ instead of $\check{\Sigma}$. This operator is essentially self-adjoint on the domain $\delta^{\frac{1}{2}}\,\mathrm{Fun}\big(\hat{\Sigma}, V_\rho^K\big)$, where $\delta^{\frac{1}{2}}$ is defined on $\hat{\Sigma}$ as in (2.11). The space $\mathrm{Fun}\big(\hat{\Sigma}, V_\rho^K\big)$, defined similarly to (2.10), consists of W-equivariant functions. We shall further elaborate on this remark, with full proofs, elsewhere.

3. Construction of twisted spin Sutherland models

Next we briefly present a general Lie theoretic framework that permits to construct a large family of spin Sutherland-type models. Then we describe some examples (which we call twisted spin Sutherland models) in detail; the full family will be studied in a future publication.

Take Y to be a compact, connected, semisimple Lie group endowed with a biinvariant Riemannian metric η induced by a multiple of the Killing form. Let G be an arbitrary *symmetric subgroup* of the product Lie group $Y \times Y$, that is, G satisfies

$$(Y \times Y)_0^\sigma \subset G \subset (Y \times Y)^\sigma, \tag{3.1}$$

where σ is an involutive automorphism of $Y \times Y$, $(Y \times Y)^\sigma$ denotes the fixed point subgroup of σ and $(Y \times Y)_0^\sigma$ is the connected component of the identity in $(Y \times Y)^\sigma$. In this general case, the following action (often called 'Hermann action')

$$\phi : G \times Y \to Y, \qquad ((a, b), y) \mapsto \phi_{(a,b)}(y) := ayb^{-1} \tag{3.2}$$

of G on Y is known to be *hyperpolar*, which means that this action is polar in such a way that the sections are *flat* in the induced metric. In fact, the sections $\Sigma \subset Y$ are provided by certain tori of Y associated with Abelian subalgebras of the correct dimension lying in the subspace $(T_e(G \cdot e))^\perp$ of T_eY. These results, and many more on related matters, can be found in [29, 30].

For example, choose an arbitrary automorphism $\Theta \in \mathrm{Aut}(Y)$ and set

$$\sigma(y_1, y_2) := (\Theta^{-1}(y_2), \Theta(y_1)) \qquad \forall (y_1, y_2) \in Y \times Y. \tag{3.3}$$

By projection to the second factor, the symmetric subgroup

$$G := \{(\Theta^{-1}(g), g) | g \in Y\} \subset Y \times Y \tag{3.4}$$

can be identified with Y, $G \cong Y$, and (3.2) then becomes the action of Y on itself by Θ-*twisted conjugations*. Indeed, after identifying G with Y, equation (3.2) yields the action whereby $g \in Y$ sends $y \in Y$ to $\Theta^{-1}(g)yg^{-1}$, which is an ordinary conjugation by g if Θ is the identity. In the most interesting cases Θ corresponds to a Dynkin diagram symmetry of Y. Some of the resulting spin Sutherland models have been investigated in our earlier work [20] at the classical level. After fixing the necessary group theoretical conventions, we describe the quantum-mechanical counterparts of these models in subsection 3.2. For simplicity, in what follows we assume that the underlying Lie group is simple and simply connected.

3.1. Conventions

Let G be a compact, connected, simply-connected, simple Lie group with fixed maximal torus $T \subset G$. Set $r := \dim(T)$. Denote by $\mathcal{A}$ and $\mathcal{H}$ the complexifications of the real Lie algebras $\mathcal{G} := \mathrm{Lie}(G)$ and $\mathcal{T} := \mathrm{Lie}(T)$. Then $\mathcal{A}$ is a complex simple Lie algebra with Cartan subalgebra $\mathcal{H}$, and we choose a polarization $\Phi = \Phi_+ \cup \Phi_-$ for the root system Φ of $(\mathcal{H}, \mathcal{A})$ and a set of simple roots $\{\varphi_k\}_{k=1}^r \subset \Phi_+$. We also select root vectors $\{X_\varphi\}_{\varphi\in\Phi}$ satisfying

$$\langle X_\varphi, X_{-\varphi}\rangle = 1 \qquad \forall \varphi \in \Phi_+, \tag{3.5}$$

where $\langle , \rangle : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ is a convenient positive multiple of the Killing form of $\mathcal{A}$. We let

$$T_{\varphi_k} := \left[X_{\varphi_k}, X_{-\varphi_k}\right] \in \mathcal{H}, \qquad 1 \leqslant k \leqslant r, \tag{3.6}$$

and thus we have

$$\mathcal{T} = \mathrm{i}\mathcal{H}_{\mathrm{r}} \qquad \text{with} \qquad \mathcal{H}_{\mathrm{r}} := \mathrm{span}_{\mathbb{R}}\left\{T_{\varphi_k} \middle| 1 \leqslant k \leqslant r\right\}. \tag{3.7}$$

By a suitable choice of the root vectors we may assume that

$$\mathcal{G} = \mathcal{T} \oplus \left(\oplus_{\varphi\in\Phi_+}\mathbb{R}Y_\varphi\right) \oplus \left(\oplus_{\varphi\in\Phi_+}\mathbb{R}Z_\varphi\right), \tag{3.8}$$

where

$$Y_\varphi := \frac{\mathrm{i}}{\sqrt{2}}(X_\varphi + X_{-\varphi}) \qquad \text{and} \qquad Z_\varphi := \frac{1}{\sqrt{2}}(X_\varphi - X_{-\varphi}) \qquad \forall\varphi\in\Phi_+. \tag{3.9}$$

The bilinear form

$$\mathcal{B} := -\langle\, ,\rangle|_{\mathcal{G}\times\mathcal{G}}: \mathcal{G}\times\mathcal{G}\to\mathbb{R} \tag{3.10}$$

is a G-invariant positive definite scalar product on $\mathcal{G}$ and (3.9) defines an orthonormal set of vectors with respect to $\mathcal{B}$. We equip G with the biinvariant Riemannian metric η induced by this scalar product.

Any symmetry θ of the Dynkin diagram of $\mathcal{A}$ extends to an automorphism of $\mathcal{A}$ by the requirement

$$\theta\left(X_{\pm\varphi_k}\right) = X_{\pm\theta(\varphi_k)} \qquad 1\leqslant k\leqslant r. \tag{3.11}$$

The resulting automorphism $\theta\in\mathrm{Aut}(\mathcal{A})$ preserves the real algebras $\mathcal{G}$ and $\mathcal{T}$ and gives rise to an automorphism Θ of the group G, which maps the torus T to itself. Then G acts on itself by the Θ-twisted conjugations mentioned before. We here designate this action as

$$I^\Theta: G\times G\to G, \qquad (g,y)\mapsto I^\Theta_g(y) := \Theta^{-1}(g)yg^{-1}. \tag{3.12}$$

A *section* for this hyperpolar action is furnished [24] by the fixed point subgroup T^Θ of Θ in the maximal torus T. The isotropy subgroup of the elements of principal I^Θ orbit type in T^Θ is given by T^Θ itself. In the notations used in proposition 2.1, we have

$$\Sigma = T^\Theta, \qquad K = T^\Theta. \tag{3.13}$$

Correspondingly, $\check{\Sigma} = \check{T}^\Theta$ stands below for a connected component of the set of I^Θ-regular elements in T^Θ. Also note that in the fixed point subalgebra $\mathcal{T}^\theta$ of θ in $\mathcal{T}$ one can choose a bounded open domain $\check{\mathcal{T}}^\theta$ (a generalized Weyl alcove) such that the exponential map restricted to $\check{\mathcal{T}}^\theta$ provides a one-to-one parametrization of $\check{T}^\Theta$. Following [24], we explain in appendix B that $\check{\mathcal{T}}^\theta$ can be characterized as the interior of a fundamental domain for the action of a 'twisted affine Weyl group' on $\mathcal{T}^\theta$. In the above we used that G is not only connected but also simply connected, since otherwise T^Θ would not be always connected [23, 24], and here we wish to avoid this complication.

From now on we assume that the automorphism (3.11) is *involutive*, and denote by $\mathcal{A}^\pm, \mathcal{G}^\pm, \mathcal{H}^\pm, \mathcal{H}^\pm_{\mathrm{r}}$ and $\mathcal{T}^\pm$ the corresponding eigensubspaces of θ with eigenvalues ±1. Then $\mathcal{A}^+$ is a complex simple Lie algebra with Cartan subalgebra $\mathcal{H}^+$, and $\mathcal{A}^-$ is an irreducible module of $\mathcal{A}^+$ whose non-zero weights have *multiplicity* 1 [31]. Moreover, $(\mathcal{T}^+,\mathcal{G}^+)$ is a compact real form of $(\mathcal{H}^+,\mathcal{A}^+)$ with the associated real irreducible module $\mathcal{G}^-$. Based on the standard 'folding procedure', detailed for example in [20], one can construct convenient bases for all these spaces from the above Weyl–Chevalley basis of $\mathcal{A}$. We next display the bases needed later.

Let $\mathfrak{R}$ be the set of roots of $(\mathcal{H}^+,\mathcal{A}^+)$ and let $\mathfrak{W}$ be the set of non-zero weights for $(\mathcal{H}^+,\mathcal{A}^-)$. We choose root vectors $X^+_\alpha(\alpha\in\mathfrak{R})$ and weight vectors $X^-_\lambda(\lambda\in\mathfrak{W})$ normalized by

$$\left\langle X^+_\alpha, X^+_{-\alpha}\right\rangle = 1 \qquad \forall \qquad \alpha\in\mathfrak{R}, \qquad \langle X^-_\lambda, X^-_{-\lambda}\rangle = 1 \qquad \forall\lambda\in\mathfrak{W}. \tag{3.14}$$

Selecting positive roots and weights, $\mathfrak{R} = \mathfrak{R}_+\cup\mathfrak{R}_-$ and $\mathfrak{W} = \mathfrak{W}_+\cup\mathfrak{W}_-$, we define

$$\begin{aligned} Y^+_\alpha &:= \frac{\mathrm{i}}{\sqrt{2}}\left(X^+_\alpha + X^+_{-\alpha}\right), \qquad & Z^+_\alpha &:= \frac{1}{\sqrt{2}}\left(X^+_\alpha - X^+_{-\alpha}\right) \qquad \forall\alpha\in\mathfrak{R}_+,\\ Y^-_\lambda &:= \frac{\mathrm{i}}{\sqrt{2}}(X^-_\lambda + X^-_{-\lambda}), \qquad & Z^-_\lambda &:= \frac{1}{\sqrt{2}}(X^-_\lambda - X^-_{-\lambda}) \qquad \forall\lambda\in\mathfrak{W}_+, \end{aligned} \tag{3.15}$$

which form an orthonormal set with respect to the scalar product $\mathcal{B}$ on $\mathcal{G}$. By using the folding procedure to construct the base elements in (3.14), we obtain the orthogonal decompositions

$$\begin{aligned}\mathcal{G}^+ &= \mathcal{T}^+ \oplus \left(\oplus_{\alpha\in\mathfrak{R}_+}\mathbb{R}Y^+_\alpha\right) \oplus \left(\oplus_{\alpha\in\mathfrak{R}_+}\mathbb{R}Z^+_\alpha\right),\\ \mathcal{G}^- &= \mathcal{T}^- \oplus \left(\oplus_{\lambda\in\mathfrak{W}_+}\mathbb{R}Y^-_\lambda\right) \oplus \left(\oplus_{\lambda\in\mathfrak{W}_+}\mathbb{R}Z^-_\lambda\right).\end{aligned} \tag{3.16}$$

As was shown at the classical level in [20], certain dynamical r-matrices enter in the description of the spin Sutherland-type models resulting from Hamiltonian reduction based on Θ-twisted conjugations. We recall that the dynamical r-matrix associated with the involutive automorphism θ is a function $R^\theta: \check{\mathcal{H}}^+ \to \mathrm{End}(\mathcal{A})$ defined on an open subset $\check{\mathcal{H}}^+ \subset \mathcal{H}^+$. Its 'shifted' versions, $R^\theta_\pm = R^\theta \pm \frac{1}{2}$, take particularly simple form, for example,

$$R^\theta_+(h) = R^\theta(h) + \frac{1}{2} = \begin{cases} \frac{1}{2} & \text{on } \mathcal{H},\\ \left(\mathbf{1} - \theta\circ \mathrm{e}^{-\mathrm{ad}_h}|_{\mathcal{H}^\perp}\right)^{-1} & \text{on } \mathcal{H}^\perp.\end{cases} \tag{3.17}$$

Take $h := \mathrm{i}q \in \check{\mathcal{T}}^+$ with some $q \in \mathcal{H}^+_{\mathrm{r}}$. Then, on the basis vectors (3.15), we have

$$\begin{aligned} &R^\theta(\mathrm{i}q)Y^+_\alpha = \frac{1}{2}\cot\left(\frac{\alpha(q)}{2}\right)Z^+_\alpha, \qquad R^\theta(\mathrm{i}q)Z^+_\alpha = -\frac{1}{2}\cot\left(\frac{\alpha(q)}{2}\right)Y^+_\alpha, \qquad \forall\alpha\in\mathfrak{R}_+,\\ &R^\theta(\mathrm{i}q)Y^-_\lambda = -\frac{1}{2}\tan\left(\frac{\lambda(q)}{2}\right)Z^-_\lambda, \quad R^\theta(\mathrm{i}q)Z^-_\lambda = \frac{1}{2}\tan\left(\frac{\lambda(q)}{2}\right)Y^-_\lambda, \qquad \forall\lambda\in\mathfrak{W}_+.\end{aligned} \tag{3.18}$$

These r-matrices, which solve the classical dynamical Yang–Baxter equation on $\mathcal{H}^+$ [32], appear in the subsequent equations.

3.2. The twisted spin Sutherland models

Now we are ready to determine the explicit form of the objects occurring in proposition 2.1 for the twisted conjugation action (3.12) based on an involutive diagram automorphism.

Take an arbitrary point $\mathrm{e}^{\mathrm{i}q} \in \check{T}^\Theta$ with $q \in \mathcal{H}^+_{\mathrm{r}}$. At this point, the value of the infinitesimal generator of the action (3.12) corresponding to $\xi \in \mathcal{G} = T_eG$ is easily seen to be

$$\xi^\sharp_{\mathrm{e}^{\mathrm{i}q}} = -(\mathrm{d}L_{\mathrm{e}^{\mathrm{i}q}})_e \circ (\mathbf{1} - \theta\circ\mathrm{e}^{-\mathrm{ad}_{\mathrm{i}q}})(\xi), \tag{3.19}$$

where $L_g: y \mapsto gy$ is the left-translation on G by $g \in G$, and we used that $\theta^{-1} = \theta$. On account of (2.14), to calculate the inertia operator we need $\xi^\sharp$ only for $\xi \in \mathcal{K}^\perp$. In our case

$$\mathcal{K} = \mathcal{T}^+, \tag{3.20}$$

and comparison with (3.17) shows that the linear isomorphism (2.13) is given by

$$\mathcal{K}^\perp \ni \xi \mapsto \xi^\sharp_{\mathrm{e}^{\mathrm{i}q}} = -(\mathrm{d}L_{\mathrm{e}^{\mathrm{i}q}})_e \circ R^\theta_+(\mathrm{i}q)^{-1}(\xi) \in T_{\mathrm{e}^{\mathrm{i}q}}G \qquad \forall \mathrm{i}q \in \check{\mathcal{T}}^+. \tag{3.21}$$

Since for the transpose of $R^\theta_+(\mathrm{i}q)$ with respect to $\mathcal{B}$ we have $R^\theta_+(\mathrm{i}q)^T = -R^\theta_-(\mathrm{i}q)$, equation (2.14) implies the following formula of the inertia operator:

$$\mathcal{J}(\mathrm{e}^{\mathrm{i}q}) = -(R^\theta_+(\mathrm{i}q)R^\theta_-(\mathrm{i}q))^{-1} \qquad \forall \mathrm{i}q \in \check{\mathcal{T}}^+. \tag{3.22}$$

After introducing an orthonormal basis $\{\mathrm{i}K^-_j\}$ in $\mathcal{T}^-$, the set of vectors $\{\mathrm{i}K^-_j, Y^+_\alpha, Z^+_\alpha, Y^-_\lambda, Z^-_\lambda\}$ is an orthonormal basis in $\mathcal{K}^\perp$, and the action of $\mathcal{J}(\mathrm{e}^{\mathrm{i}q})$ on these vectors spells as

$$\begin{aligned} &\mathcal{J}(\mathrm{e}^{\mathrm{i}q})\mathrm{i}K^-_j = 4\mathrm{i}K^-_j,\\ &\mathcal{J}(\mathrm{e}^{\mathrm{i}q})Y^+_\alpha = 4\sin^2\left(\frac{\alpha(q)}{2}\right)Y^+_\alpha, \qquad \mathcal{J}(\mathrm{e}^{\mathrm{i}q})Z^+_\alpha = 4\sin^2\left(\frac{\alpha(q)}{2}\right)Z^+_\alpha,\\ &\mathcal{J}(\mathrm{e}^{\mathrm{i}q})Y^-_\lambda = 4\cos^2\left(\frac{\lambda(q)}{2}\right)Y^-_\lambda, \qquad \mathcal{J}(\mathrm{e}^{\mathrm{i}q})Z^-_\lambda = 4\cos^2\left(\frac{\lambda(q)}{2}\right)Z^-_\lambda.\end{aligned} \tag{3.23}$$

These relations come from (3.18) together with the identity $R_+^\theta R_-^\theta = (R^\theta)^2 - \frac{1}{4}$. Since the matrix of $\mathcal{J}(\mathrm{e}^{\mathrm{i}q})$ in the above basis is diagonal, it is easy to find its inverse and determinant. The non-zero entries of the matrix of $\mathcal{J}(\mathrm{e}^{\mathrm{i}q})^{-1}$ appear in the list

$$\begin{aligned}
&\mathcal{B}(\mathrm{i}K_k^-, \mathcal{J}(\mathrm{e}^{\mathrm{i}q})^{-1}\mathrm{i}K_l^-) = \frac{1}{4}\delta_{k,l},\\
&\mathcal{B}\big(Y_\alpha^+, \mathcal{J}(\mathrm{e}^{\mathrm{i}q})^{-1}Y_\beta^+\big) = \frac{1}{4\sin^2\left(\frac{\alpha(q)}{2}\right)}\delta_{\alpha,\beta}, \qquad \mathcal{B}\big(Z_\alpha^+, \mathcal{J}(\mathrm{e}^{\mathrm{i}q})^{-1}Z_\beta^+\big) = \frac{1}{4\sin^2\left(\frac{\alpha(q)}{2}\right)}\delta_{\alpha,\beta},\\
&\mathcal{B}(Y_\lambda^-, \mathcal{J}(\mathrm{e}^{\mathrm{i}q})^{-1}Y_\mu^-) = \frac{1}{4\cos^2\left(\frac{\lambda(q)}{2}\right)}\delta_{\lambda,\mu}, \qquad \mathcal{B}(Z_\lambda^-, \mathcal{J}(\mathrm{e}^{\mathrm{i}q})^{-1}Z_\mu^-) = \frac{1}{4\cos^2\left(\frac{\lambda(q)}{2}\right)}\delta_{\lambda,\mu}.
\end{aligned} \tag{3.24}$$

These formulae directly give the third term in the reduced Laplace–Beltrami operator (2.17).

To calculate the second term in (2.17), we first observe from (2.16) and the above that, up to an irrelevant non-zero multiplicative constant (whose sign depends on the choice of $\check{\mathcal{T}}^+$), the density function now has the form

$$\delta^{\frac{1}{2}}(\mathrm{e}^{\mathrm{i}q}) = \prod_{\alpha\in\mathfrak{R}_+} \sin\left(\frac{\alpha(q)}{2}\right) \prod_{\lambda\in\mathfrak{W}_+} \cos\left(\frac{\lambda(q)}{2}\right). \tag{3.25}$$

Choosing an orthonormal basis $\{\mathrm{i}K_j^+\}$ in $\mathcal{T}^+$, we parametrize $\check{T}^\Theta$ by the diffeomorphism

$$\check{\mathcal{T}}^+ \ni \mathrm{i}q = q^k \mathrm{i}K_k^+ \mapsto \mathrm{e}^{\mathrm{i}q} \in \check{T}^\Theta, \tag{3.26}$$

where $\check{\mathcal{T}}^+$ is an appropriate bounded open domain in $\mathcal{T}^+$. In the coordinates $\{q^k\}$ the Laplace–Beltrami operator $\Delta_{\check{T}^\Theta}$ is simply[6]

$$\Delta_{\check{T}^\Theta} = \sum_k \partial_k^2, \tag{3.27}$$

where $\partial_k := \frac{\partial}{\partial q^k}$. We then find that in the present case the second term of (2.17) yields just a constant,

$$\big(\delta^{-\frac{1}{2}}\Delta_{\check{T}^\Theta}\big(\delta^{\frac{1}{2}}\big)\big)(\mathrm{e}^{\mathrm{i}q}) = -\langle \varrho_{\mathcal{G}}^\theta, \varrho_{\mathcal{G}}^\theta\rangle \tag{3.28}$$

with

$$\varrho_{\mathcal{G}}^\theta := \frac{1}{2}\sum_{\alpha\in\mathfrak{R}_+}\alpha + \frac{1}{2}\sum_{\lambda\in\mathfrak{W}_+}\lambda, \tag{3.29}$$

generalizing the well-known $\theta = \mathrm{id}$ case [14, 16]. The derivation of (3.28) and the value of $\langle \varrho_{\mathcal{G}}^\theta, \varrho_{\mathcal{G}}^\theta\rangle$ are elaborated in appendix A. The foregoing considerations are now summarized by the following result.

Proposition 3.1. *Consider the twisted conjugation action (3.12) of a compact, connected, simply connected, simple Lie group G on itself, equipped with the Riemannian metric corresponding to the scalar product $\mathcal{B}$ (3.10) on $\mathcal{G}$. Then, using the above notations, the reduced Laplace–Beltrami operator (2.17) associated with a finite-dimensional unitary irrep (ρ, V_ρ) of G takes the form*

$$\begin{aligned}
\Delta_{\mathrm{red}} = \Delta_{\check{T}^\Theta} &+ \langle \varrho_{\mathcal{G}}^\theta, \varrho_{\mathcal{G}}^\theta\rangle + \frac{1}{4}\sum_j \rho'(\mathrm{i}K_j^-)^2\\
&+ \frac{1}{4}\sum_{\alpha\in\mathfrak{R}_+}\frac{\rho'\big(Y_\alpha^+\big)^2 + \rho'\big(Z_\alpha^+\big)^2}{\sin^2\left(\frac{\alpha(q)}{2}\right)} + \frac{1}{4}\sum_{\lambda\in\mathfrak{W}_+}\frac{\rho'(Y_\lambda^-)^2 + \rho'(Z_\lambda^-)^2}{\cos^2\left(\frac{\lambda(q)}{2}\right)}.
\end{aligned} \tag{3.30}$$

[6] The coordinates q^k associated with an orthonormal basis of $\mathcal{T}^+$ should not be confused with the components q_i used in the examples presented at the end of this section and in section 5.

This operator is essentially self-adjoint on the dense domain $\delta^{\frac{1}{2}}\,\mathrm{Fun}\big(\check{T}^{\Theta}, V_\rho^K\big)$ of the reduced Hilbert space $L^2\big(\check{T}^{\Theta}, V_\rho^K, \mathrm{d}\mu_{\check{T}^\Theta}\big)$, where $K = T^\Theta$ and $\check{T}^\Theta$ is parametrized according to (3.26).

Now we have a few remarks to make. First, since $K = T^\Theta$ is connected, we can equivalently write V_ρ^K as $V_\rho^{\mathcal{T}^+}$, which denotes the set of vectors annihilated by $\mathcal{T}^+$ in the representation ρ'. We shall make use of this remark in section 5. Second, as follows for example by comparison with the results in [24], the possible domains $\check{\mathcal{T}}^+ \subset \mathcal{T}^+$ that parametrize $\check{T}^\Theta$ in a one-to-one manner by the exponential map can be determined very explicitly from formula (3.25). Namely, each connected component of the open set $\{\mathrm{i}q\,|\,\mathrm{i}q \in \mathcal{T}^+, \delta(\mathrm{e}^{\mathrm{i}q}) > 0\} \subset \mathcal{T}^+$ is an appropriate choice for $\check{\mathcal{T}}^+$. Using the coordinates q^k (3.26), $\mathrm{d}\mu_{\check{T}^\Theta} = \prod_k \mathrm{d}q^k$ is the Lebesgue measure on $\check{\mathcal{T}}^+$. Third, according to remark 2.3, one can also describe the reduced system in terms of the larger configuration space $\hat{T}^\Theta$. In this description, the wavefunctions are equivariant with respect to the generalized Weyl group (2.18), which in the present case becomes the so-called twisted Weyl group

$$W(G, T^\Theta, \Theta) := N_G(T^\Theta, I^\Theta)/T^\Theta, \tag{3.31}$$

where

$$N_G(T^\Theta, I^\Theta) = \big\{g \in G\,\big|\,I_g^\Theta(T^\Theta) = T^\Theta\big\}. \tag{3.32}$$

Based on [22–24], we review the structure of this group in appendix B.

We finish this section by listing the positive roots $\mathfrak{R}_+$ and weights $\mathfrak{W}_+$ and a possible choice for $\check{\mathcal{T}}^+ \subset \mathcal{T}^+$ for the non-trivial involutive diagram automorphisms of the classical Lie algebras.

If $\mathcal{A} = D_{n+1}$, then $\mathcal{A}^+ = B_n$ and the module $\mathcal{A}^-$ is isomorphic to the defining representation of B_n. The real Abelian subalgebra $\mathcal{H}_\mathrm{r}^+$ can be realized as real diagonal matrices of the form

$$q = \mathrm{diag}(q_1, q_2, \ldots, q_n, 0, 0, -q_n, \ldots, -q_2, -q_1). \tag{3.33}$$

By introducing the linear functionals $e_m : q \mapsto q_m$, we can write

$$\mathfrak{R}_+ = \{e_k \pm e_l, e_m \,|\, 1 \leqslant k < l \leqslant n, 1 \leqslant m \leqslant n\}, \qquad \mathfrak{W}_+ = \{e_m \,|\, 1 \leqslant m \leqslant n\}. \tag{3.34}$$

A possible choice for $\check{\mathcal{T}}^+ \subset \mathcal{T}^+$ is given by the set

$$\check{\mathcal{T}}^+ = \{\mathrm{i}q \,|\, 0 < q_n < q_{n-1} < \cdots < q_2 < q_1 < \pi\}. \tag{3.35}$$

If $\mathcal{A} = A_{2n-1}$, then $\mathcal{A}^+ = C_n$ and $\mathcal{H}_\mathrm{r}^+$ can be realized as diagonal matrices of the form

$$q = \mathrm{diag}(q_1, q_2, \ldots, q_n, -q_n, \ldots, -q_2, -q_1). \tag{3.36}$$

Using the functionals $e_m : q \mapsto q_m$, we can write

$$\mathfrak{R}_+ = \{e_k \pm e_l, 2e_m \,|\, 1 \leqslant k < l \leqslant n, 1 \leqslant m \leqslant n\}, \qquad \mathfrak{W}_+ = \{e_k \pm e_l \,|\, 1 \leqslant k < l \leqslant n\}. \tag{3.37}$$

For $\check{\mathcal{T}}^+$ we can choose the set

$$\check{\mathcal{T}}^+ = \{\mathrm{i}q \,|\, 0 < q_n < q_{n-1} < \cdots < q_2 < q_1 < \pi, 0 < q_1 + q_2 < \pi\}. \tag{3.38}$$

For $\mathcal{A} = A_{2n}$ one has $\mathcal{A}^+ = B_n$. Now $\mathcal{H}_\mathrm{r}^+$ can be realized as

$$q = \mathrm{diag}(q_1, q_2, \ldots, q_n, 0, -q_n, \ldots, -q_2, -q_1), \tag{3.39}$$

and with $e_m : q \mapsto q_m$ we have

$$\begin{aligned} \mathfrak{R}_+ &= \{e_k \pm e_l, e_m \,|\, 1 \leqslant k < l \leqslant n, 1 \leqslant m \leqslant n\}, \\ \mathfrak{W}_+ &= \{e_k \pm e_l, e_m, 2e_m \,|\, 1 \leqslant k < l \leqslant n, 1 \leqslant m \leqslant n\}. \end{aligned} \tag{3.40}$$

A possible choice for $\check{\mathcal{T}}^+$ is the bounded open domain

$$\check{\mathcal{T}}^+ = \left\{\mathrm{i}q \,|\, 0 < q_n < q_{n-1} < \cdots < q_2 < q_1 < \frac{\pi}{2}\right\}. \tag{3.41}$$

4. How to diagonalize the reduced Hamiltonians?

We explain below how the diagonalization of the 'twisted spin Sutherland Hamiltonians' Δ_{red} given by proposition 3.1 can be performed in principle in terms of certain Clebsch–Gordan problems in the representation theory of the underlying symmetry group G.

Let us start by noting that the Hilbert space $L^2(G, \mathrm{d}\mu_G)$ carries the unitary representation $U_\theta(g)$ of the group G defined by

$$U_\theta(g)\psi := \psi \circ I^\Theta_{g^{-1}} \tag{4.1}$$

for any 'wavefunction' ψ and $g \in G$. Let L^+ be the set of dominant integral weights of G and for any $\Lambda \in L^+$ denote by $(\rho_\Lambda, V_\Lambda)$ the corresponding irreducible unitary highest weight representation[7], of dimension d_Λ. By arbitrarily fixing a basis in V_Λ, introduce the matrix elements $\rho_\Lambda^{a,b} \in L^2(G, \mathrm{d}\mu_G)$ spanning the subspace

$$F^\Lambda := \mathrm{span}\big\{\rho_\Lambda^{a,b}\big| a, b = 1, \ldots, d_\Lambda\big\} \tag{4.2}$$

of $L^2(G, \mathrm{d}\mu_G)$. By the Peter–Weyl theorem, one has the orthogonal direct sum decomposition

$$L^2(G, \mathrm{d}\mu_G) = \oplus_{\Lambda\in L^+} F^\Lambda. \tag{4.3}$$

The restriction of the representation U_θ to F^Λ can be written as the following tensor product of irreps:

$$F^\Lambda = V_{(\Lambda\circ\theta^{-1})^*} \otimes V_\Lambda, \tag{4.4}$$

where Λ^* denotes the highest weight of the complex conjugate of the representation ρ_Λ and, as θ gives a Cartan-preserving automorphism of the complexification of $\mathcal{G}$, $\Lambda \circ \theta^{-1} \in L^+$ is a well-defined functional on the Cartan subalgebra. With these notations, using the fact that θ is an involution, we have

$$L^2(G, \mathrm{d}\mu_G) = \oplus_{\Lambda\in L^+} V_{(\Lambda\circ\theta)^*} \otimes V_\Lambda. \tag{4.5}$$

Focusing now on our Hamiltonian reduction problem, let us set

$$(\rho, V_\rho) := (\rho_\nu, V_\nu) \tag{4.6}$$

with some highest weights ν (see footnote 7). Then the reduced Hilbert space takes the form

$$L^2(G, V_\nu, \mathrm{d}\mu_G)^G = \oplus_{\Lambda\in L^+} (V_{(\Lambda\circ\theta)^*} \otimes V_\Lambda \otimes V_\nu)^G. \tag{4.7}$$

Therefore we have to find the singlets, i.e., the G-invariant states, in the threefold tensor products in (4.7). Any such singlet state, say

$$\mathcal{F}_{\Lambda,\nu} \in (V_{(\Lambda\circ\theta)^*} \otimes V_\Lambda \otimes V_\nu)^G, \tag{4.8}$$

is in effect a V_ν-valued I^Θ-equivariant function on G.

The point is that *the functions $\mathcal{F}_{\Lambda,\nu}$ are eigenstates of the Laplace–Beltrami operator Δ_G*,

$$\Delta_G \mathcal{F}_{\Lambda,\nu} = -\langle \Lambda + 2\varrho_{\mathcal{G}}, \Lambda\rangle \mathcal{F}_{\Lambda,\nu}, \tag{4.9}$$

since Δ_G corresponds to the quadratic Casimir operator of the Lie algebra $\mathcal{G}$ that survives the reduction. $\varrho_{\mathcal{G}}$ stands for the sum of the fundamental weights of the complexification, $\mathcal{A}$, of $\mathcal{G}$. More generally, take any left G-invariant, formally self-adjoint scalar differential operator over G induced by a corresponding element of the center of the universal enveloping algebra of $\mathcal{A}$. Any such differential operator, say $\hat{P}$, takes a constant value on F^Λ,

$$\hat{P}\mathcal{F} = p(\Lambda + \varrho_{\mathcal{G}})\mathcal{F} \qquad \forall \mathcal{F} \in F^\Lambda, \tag{4.10}$$

[7] We should have written V_{ρ_Λ} according to our general notation used before, but here we simplify this to V_Λ.

with an associated Weyl invariant polynomial function p on the dual of the Cartan subalgebra. The map sending $\hat{P}$ to p is the so-called Harish–Chandra isomorphism [33]. (A rather explicit formula for the eigenvalue $p(\Lambda + \varrho_{\mathcal{G}})$ of the Casimir operator $\hat{P}$ is contained in [34].) It is standard to show that these commuting differential operators induce commuting self-adjoint 'Hamiltonians' for the reduced quantum system.

Returning to (4.7), one has the Clebsch–Gordan series

$$V_\Lambda \otimes V_\nu = \oplus_{\lambda \in L^+} N^\lambda_{\Lambda,\nu} V_\lambda, \tag{4.11}$$

where $N^\lambda_{\Lambda,\nu} \in \mathbb{Z}_+$ are often called Littlewood–Richardson numbers. In principle, they can be found algorithmically since the irreducible characters are known. In order to find the spectra of the reduced Laplace–Beltrami operator (as well as the spectra of the higher Casimirs) on $L^2(G, V_\nu, d\mu_G)^G$, we need the numbers

$$\dim(V_{(\Lambda\circ\theta)^*} \otimes V_\Lambda \otimes V_\nu)^G = N^{\Lambda\circ\theta}_{\Lambda,\nu}. \tag{4.12}$$

In fact, it follows from the definition of Δ_{red} that

$$\text{spectrum}\,(\Delta_{\mathrm{red}}) = \left\{ -\langle \Lambda + 2\varrho_{\mathcal{G}}, \Lambda\rangle \,\middle|\, \Lambda \in L^+,\ N^{\Lambda\circ\theta}_{\Lambda,\nu} \neq 0 \right\}. \tag{4.13}$$

The Littlewood–Richardson numbers determine the multiplicities of the eigenvalues of Δ_{red} as well. It is an archetypical Clebsch–Gordan problem to find *explicitly* the pairs (Λ, ν) for which $N^{\Lambda\circ\theta}_{\Lambda,\nu} \neq 0$. It is even more difficult to obtain the explicit form of the states $\mathcal{F}_{\Lambda,\nu}$, since they involve the Clebsch–Gordan coefficients themselves. The diagonalization of the reduced Hamiltonian boils down to these finite-dimensional, purely algebraic problems.

The eigenfunctions $\mathcal{F}_{\Lambda,\nu}$ 'descend' to certain trigonometric polynomials through the reduction procedure. To see this, let us take a basis v^Λ_a in V_Λ consisting of weight vectors with respect to $\mathcal{H}^+$, with weight μ_a:

$$\rho'_\Lambda(\mathrm{i}q)v^\Lambda_a = \mathrm{i}\mu_a(q)v^\Lambda_a \qquad \forall q \in \mathcal{H}^+_{\mathrm{r}}. \tag{4.14}$$

Similarly, take a basis u^ν_k in $V_\nu^{T^\Theta}$. The restricted eigenfunction of Δ_G,

$$\hat{f}_{\Lambda,\nu} := \mathcal{F}_{\Lambda,\nu}|_{T^\Theta}, \tag{4.15}$$

is a linear combination of functions of the form

$$\left(v^\Lambda_a, \rho_\Lambda(\mathrm{e}^{\mathrm{i}q})v^\Lambda_b\right)u^\nu_k, \tag{4.16}$$

where $\mu_a + \mu_b = 0$. Therefore we can write

$$\hat{f}_{\Lambda,\nu}(\mathrm{e}^{\mathrm{i}q}) = \sum_{a,k} C_{a,k}\,\mathrm{e}^{\mathrm{i}\mu_a(q)}u^\nu_k \tag{4.17}$$

with some constants $C_{a,k}$. The sum runs over all the $\mathcal{H}^+$-weights of the representation space V_Λ and a basis of $V_\nu^{T^\Theta}$. It is also worth noting that, as a result of the G-equivariance of $\mathcal{F}_{\Lambda,\nu}$, the $\hat{f}_{\Lambda,\nu}$ are Weyl-equivariant functions on T^Θ, where the relevant generalized Weyl group is $W(G, T^\Theta, \Theta)$ (3.31), which acts naturally both on $V_\nu^{T^\Theta}$ and T^Θ. Of course, the eigenfunctions of the spin Sutherland operator Δ_{red}, obtained as a similarity transform of the restriction of Δ_G, also include the pre-factor $\delta^{\frac{1}{2}}$ (3.25) in front of the above function $\hat{f}_{\Lambda,\nu}$ or rather in front of $f_{\Lambda,\nu} := \hat{f}_{\Lambda,\nu}|\check{T}^\Theta$. See also the remarks after proposition 2.1 and appendix B.

5. Some examples with explicitly computable spectra

In a few distinguished cases the Littlewood–Richardson numbers (4.12) can be determined explicitly due to 'standard plethysm' rules in representation theory [35]. For $\mathcal{G} = su(N)$, such are the cases

$$\nu = k\lambda_1 \quad \text{or} \quad k\lambda_{N-1} \quad \text{and} \quad \nu = \lambda_a \quad \text{for} \quad a = 2, \ldots, (N-2), \tag{5.1}$$

where from now on we denote by λ_a, $a = 1, \ldots, \mathrm{rank}(\mathcal{G})$, the fundamental weights. Indeed, in these cases one has the so-called Pieri formulae for the decomposition of the tensor products $V_\Lambda \otimes V_\nu$. The first two cases are essentially equivalent, being related by the Dynkin diagram symmetry of A_{N-1}. Below we concentrate on the first case, since it also contains an arbitrary parameter $k \in \mathbb{Z}_+$. Throughout this section, we use $\varrho_{su(N)} = \sum_{a=1}^{N-1} \lambda_a$, which is the $\theta = \mathrm{id}$ special case of $\varrho^\theta_{su(N)}$ in (3.29). We let $T_N \subset SU(N)$, $\mathcal{T}_N \subset su(N)$ stand for the standard maximal torus and its Lie algebra, and denote the root system of $sl(N, \mathbb{C})$ by Φ^N. We fix the invariant bilinear form on $sl(N, \mathbb{C})$ to be

$$\langle X, Y \rangle := \mathrm{tr}(XY) \qquad \forall X, Y \in sl(N, \mathbb{C}). \tag{5.2}$$

5.1. *Oscillator realization of $V_{k\lambda_1}$ for $su(N)$ and corresponding models*

In this subsection, we recall the oscillator realization of $V_{k\lambda_1}$ and spell out the corresponding models both for the trivial and the non-trivial Dynkin diagram automorphisms of $su(N)$.

Let us introduce N pairs of bosonic annihilation and creation operators, a_i and $a_i^\dagger$,

$$\left[a_i, a_j^\dagger\right] = \delta_{i,j} \qquad (1 \leqslant i, j \leqslant N). \tag{5.3}$$

These operators act on the Fock space, $\mathcal{V}$, spanned by the basis

$$\prod_{i=1}^N \left(a_i^\dagger\right)^{n_i} v_0 \qquad (\forall n_i \in \mathbb{Z}_+), \tag{5.4}$$

where v_0 is the vacuum vector annihilated by a_i. We also need the 'number operators'

$$\hat{n}_i := a_i^\dagger a_i, \tag{5.5}$$

whose eigenvalues are n_i in (5.4). The Lie algebra $gl(N, \mathbb{C})$ is represented on $\mathcal{V}$ by

$$E_{i,j} \mapsto \rho'(E_{i,j}) := a_i^\dagger a_j. \tag{5.6}$$

One has $\mathcal{V} = \oplus_{k \in \mathbb{Z}_+} \mathcal{V}_k$, where $\mathcal{V}_k$ is the eigensubspace of the 'total number operator' $\left(\sum_{i=1}^N \hat{n}_i\right)$ with eigenvalue k, which is an invariant subspace for $gl(N, \mathbb{C})$. Restriction of ρ' to $\mathcal{V}_k$, and to the subalgebra $su(N) \subset gl(N, \mathbb{C})$, yields a model of the highest weight representation $V_{k\lambda_1}$. We display this fact symbolically as

$$V_{k\lambda_1} \cong \mathcal{V}_k. \tag{5.7}$$

It is easy to see from (5.6) that $\mathcal{V}_k^{\mathcal{T}_N}$ is non-zero if and only if $k = \gamma N$ for some $\gamma \in \mathbb{Z}_+$ and

$$\mathcal{V}_{\gamma N}^{\mathcal{T}_N} = \mathbb{C} \prod_{i=1}^N \left(a_i^\dagger\right)^\gamma v_0 \tag{5.8}$$

is one dimensional. It also follows from (5.6) that

$$(\rho'(E_{i,j})\rho'(E_{j,i}))|_{\mathcal{V}_{\gamma N}^{\mathcal{T}_N}} = \gamma(\gamma+1)\,\mathrm{id}_{\mathcal{V}_{\gamma N}^{\mathcal{T}_N}} \qquad \forall i \neq j. \tag{5.9}$$

Thus in the non-twisted case (with $\mathcal{G} = su(N)$, $\theta = \mathrm{id}$ and $V = V_{k\lambda_1}$), Δ_{red} in (3.30) yields

$$\Delta^\gamma_{\mathrm{red}} = \Delta_{\check{T}_N} + \langle \varrho_{su(N)}, \varrho_{su(N)} \rangle - \frac{1}{2} \sum_{\alpha \in \Phi^N_+} \frac{\gamma(\gamma+1)}{\sin^2 \alpha\left(\frac{q}{2}\right)}, \tag{5.10}$$

which is the standard Sutherland operator with coupling constant determined by $\gamma \in \mathbb{Z}_+$ ($\check{T}_N$ is parametrized by $\mathrm{e}^{\mathrm{i}q}$ with q in a Weyl alcove). This result is of course well known [10, 15].

Turning to the twisted spin Sutherland models, first note that the non-trivial automorphism of $su(N)$ can be defined on the generators $E_{i,j} \in gl(N, \mathbb{C})$ as

$$\theta(E_{i,j}) := -(-1)^{i+j} E_{N+1-j,N+1-i}. \tag{5.11}$$

Combining this with (5.6), we obtain that if $N = 2n$ is even, then

$$\mathcal{V}_k^{\mathcal{T}_{2n}^+} = \text{span}\left\{ \prod_{i=1}^{n} \left(a_i^\dagger a_{2n+1-i}^\dagger\right)^{n_i} v_0 \middle| 2\sum_{i=1}^{n} n_i = k \right\}. \tag{5.12}$$

The non-triviality of this space requires k to be even, and

$$\dim\left(\mathcal{V}_{2\kappa}^{\mathcal{T}_{2n}^+}\right) = \binom{\kappa+n-1}{n-1}, \qquad \forall \kappa, n \in \mathbb{Z}_+, n \geqslant 2. \tag{5.13}$$

If $N = (2n+1)$ is odd, then the index $i = (n+1)$ is special and we introduce the notation

$$c := a_{n+1}, \qquad c^\dagger = a_{n+1}^\dagger, \qquad \hat{m} := c^\dagger c. \tag{5.14}$$

In this case we find that

$$\mathcal{V}_k^{\mathcal{T}_{2n+1}^+} = \text{span}\left\{ (c^\dagger)^m \prod_{i=1}^{n} \left(a_i^\dagger a_{2n+2-i}^\dagger\right)^{n_i} v_0 \middle| m + 2\sum_{i=1}^{n} n_i = k \right\}, \tag{5.15}$$

and

$$\dim\left(\mathcal{V}_k^{\mathcal{T}_{2n+1}^+}\right) = \sum_{\kappa=0}^{[k/2]} \binom{\kappa+n-1}{n-1}, \qquad \forall k, n \in \mathbb{Z}_+, n \geqslant 1. \tag{5.16}$$

This dimension formula was obtained by setting $2\kappa := 2\sum_{i=1}^{n} n_i = k - m$.

Based on (5.6), it is a matter of straightforward calculation to derive from proposition 3.1 explicit formulae for the reduced Laplace–Beltrami operators in the above cases. We give a brief outline of this calculation in appendix C, and here only display the result. For this purpose, for any $N = 2n$ or $N = (2n+1)$, we define the operators

$$\begin{aligned} &\mathcal{A}_{i,j} := 2\hat{n}_i\hat{n}_j + \hat{n}_i + \hat{n}_j, \qquad 1 \leqslant i \leqslant j \leqslant n, \\ &\mathcal{B}_{i,j} := (-1)^{i+j}\left(a_i^\dagger a_{N+1-i}^\dagger a_j a_{N+1-j} + a_i a_{N+1-i} a_j^\dagger a_{N+1-j}^\dagger\right), \qquad 1 \leqslant i < j \leqslant n. \end{aligned} \tag{5.17}$$

If $N = (2n+1)$, then using (5.14) we also define

$$\begin{aligned} &\mathcal{C}_i := 2\hat{n}_i\hat{m} + \hat{m} + \hat{n}_i, \qquad 1 \leqslant i \leqslant n, \\ &\mathcal{D}_i := (-1)^{i+n}\left(a_i^\dagger a_{2n+2-i}^\dagger cc + a_i a_{2n+2-i} c^\dagger c^\dagger\right), \qquad 1 \leqslant i \leqslant n. \end{aligned} \tag{5.18}$$

These operators act on the space $\mathcal{V}_k^{\mathcal{T}_N^+}$, where the reduced wavefunctions take their values.

Proposition 5.1. *Choosing $N = 2n$ ($n \geqslant 2$) even and the representation $V_{2\kappa\lambda_1} \cong \mathcal{V}_{2\kappa}$ ($\kappa \in \mathbb{Z}_+$) of $G = SU(N)$, the reduced Laplace–Beltrami operator (3.30) corresponding to the diagram automorphism (5.11) is given by the explicit formula*

$$\begin{aligned} \Delta_{\text{red}} = {} & \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial q_i^2} + \left\langle \varrho_{su(2n)}^\theta, \varrho_{su(2n)}^\theta \right\rangle + \frac{\kappa^2}{2n} - \frac{1}{2}\sum_{i=1}^{n}\hat{n}_i^2 - \frac{1}{4}\sum_{i=1}^{n}\frac{\mathcal{A}_{i,i}}{\sin^2(q_i)} \\ & - \frac{1}{4}\sum_{1\leqslant i<j\leqslant n}\left[\frac{\mathcal{A}_{i,j}-\mathcal{B}_{i,j}}{\sin^2\left(\frac{q_i-q_j}{2}\right)} + \frac{\mathcal{A}_{i,j}+\mathcal{B}_{i,j}}{\cos^2\left(\frac{q_i-q_j}{2}\right)} + \frac{\mathcal{A}_{i,j}+\mathcal{B}_{i,j}}{\sin^2\left(\frac{q_i+q_j}{2}\right)} + \frac{\mathcal{A}_{i,j}-\mathcal{B}_{i,j}}{\cos^2\left(\frac{q_i+q_j}{2}\right)}\right], \end{aligned} \tag{5.19}$$

where we use (5.17). The constant $\left\langle \varrho_{su(2n)}^\theta, \varrho_{su(2n)}^\theta \right\rangle$ is given by (A.7) in appendix A, and the coordinates q_i can be taken to vary in the domain (3.38).

Proposition 5.2. *Choosing $N = (2n+1)$ $(n \geqslant 1)$ odd and the representation $V_{k\lambda_1} \cong \mathcal{V}_k$ $(k \in \mathbb{Z}_+)$ of $G = SU(N)$, the reduced Laplace–Beltrami operator (3.30) corresponding to the diagram automorphism (5.11) is given by the explicit formula*

$$\begin{aligned}\Delta_{\mathrm{red}} = {} & \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial q_i^2} + \left\langle \varrho^{\theta}_{su(2n+1)}, \varrho^{\theta}_{su(2n+1)} \right\rangle - \frac{1}{2}\sum_{i=1}^{n}(\hat{n}_i - \hat{m})^2 \\ & + \frac{1}{2n}\left[\sum_{i=1}^{n}(\hat{n}_i - \hat{m})\right]^2 - \frac{1}{4}\sum_{i=1}^{n}\left[\frac{\mathcal{C}_i + \mathcal{D}_i}{\sin^2\left(\frac{q_i}{2}\right)} + \frac{\mathcal{C}_i - \mathcal{D}_i}{\cos^2\left(\frac{q_i}{2}\right)} + \frac{\mathcal{A}_{i,i}}{\cos^2(q_i)}\right] \\ & - \frac{1}{4}\sum_{1\leqslant i<j\leqslant n}\left[\frac{\mathcal{A}_{i,j} - \mathcal{B}_{i,j}}{\sin^2\left(\frac{q_i-q_j}{2}\right)} + \frac{\mathcal{A}_{i,j} + \mathcal{B}_{i,j}}{\cos^2\left(\frac{q_i-q_j}{2}\right)} + \frac{\mathcal{A}_{i,j} - \mathcal{B}_{i,j}}{\sin^2\left(\frac{q_i+q_j}{2}\right)} + \frac{\mathcal{A}_{i,j} + \mathcal{B}_{i,j}}{\cos^2\left(\frac{q_i+q_j}{2}\right)}\right],\end{aligned} \tag{5.20}$$

where we use the operators defined in (5.17), (5.18). The constant $\left\langle \varrho^{\theta}_{su(2n+1)}, \varrho^{\theta}_{su(2n+1)} \right\rangle$ is evaluated in (A.7) and now the coordinates q_i can be taken to vary in the domain (3.41).

The above formulae of Δ_{red} can be somewhat simplified by using obvious trigonometry, like $4\sin^2\frac{x}{2}\cos^2\frac{x}{2} = \sin^2 x$. The operators $\mathcal{A}_{i,j}, \mathcal{B}_{i,j}, \mathcal{C}_i$ and $\mathcal{D}_i$ act on the respective spaces $\mathcal{V}_k^{\mathcal{T}_N^+}$, where $\mathcal{A}_{i,j}$ and $\mathcal{C}_i$ are diagonal, whilst $\mathcal{B}_{i,j}$ and $\mathcal{D}_i$ are off-diagonal in the natural bases that appear in equations (5.12) and (5.15). Subsequently, we shall determine the spectra of the self-adjoint matrix differential operators Δ_{red} given by (5.19) and (5.20).

5.2. Determination of the spectra from the Pieri formula

For $\mathcal{G} = su(N)$, write an arbitrary $\Lambda \in L^+$ in the form

$$\Lambda = \sum_{i=1}^{N-1} M^i\lambda_i \qquad M^i \in \mathbb{Z}_+. \tag{5.21}$$

Then the so-called *Pieri formula* (see, e.g., [35]) says that precisely those

$$\lambda = \sum_{i=1}^{N-1} m^i\lambda_i \qquad m^i \in \mathbb{Z}_+ \tag{5.22}$$

appear in the tensor product $V_\Lambda \otimes V_{k\lambda_1}$ for which

$$m^i = M^i + C^i - C^{i+1} \qquad \forall i = 1, \ldots, (N-1) \tag{5.23}$$

with some $(C^1, C^2, \ldots, C^{N-1}, C^N) \in \mathbb{Z}_+^N$ such that

$$C^{i+1} \leqslant M^i \qquad \forall i = 1, \ldots, (N-1), \qquad \sum_{i=1}^{N} C^i = k. \tag{5.24}$$

Note that $N^{\lambda}_{\Lambda,k\lambda_1} = 1$ for all λ for which $N^{\lambda}_{\Lambda,k\lambda_1} \neq 0$. The spectra of the reduced Laplace–Beltrami operators displayed in subsection 5.1 are easily found by utilizing the Pieri formula.

5.2.1. The standard Sutherland model with integer couplings.

In the non-twisted $su(N)$ case, for $\theta = \mathrm{id}$ and $\nu = k\lambda_1$, the determination of the spectrum (4.13) boils down to finding all $\Lambda \in L^+$ for which

$$N^{\Lambda}_{\Lambda,k\lambda_1} = 1. \tag{5.25}$$

From the Pieri formula (5.23) with $m^i = M^i$ we immediately get that

$$C^1 = C^2 = \cdots = C^N := \gamma \in \mathbb{Z}_+ \tag{5.26}$$

and therefore $k = \gamma N$ with some $\gamma \in \mathbb{Z}_+$, which also follows from the condition $\dim\big(V^{\mathcal{T}_N}_{k\lambda_1}\big) \neq 0$ as we saw. The first part of (5.24) then says that we must have

$$M^i = \gamma + \mu^i \qquad \text{with some arbitrary} \quad \mu^i \in \mathbb{Z}_+. \tag{5.27}$$

This means that *the admissible weights* Λ *are precisely the weights of the form*

$$\Lambda = \gamma \varrho_{su(N)} + \mu, \qquad \forall \mu = \sum_{i=1}^{N-1} \mu^i \lambda_i \in L^+. \tag{5.28}$$

We have already seen that $\dim\big(V^{\mathcal{T}_N}_{\gamma N\lambda_1}\big) = 1$ by (5.8), and have derived formula (5.10) of Δ_{red} that arises in this case. We now rewrite this scalar Schrödinger operator in the form

$$\Delta^{\gamma}_{\text{red}} = \frac{1}{4}\sum_{i=1}^{N-1}\left(\frac{\partial}{\partial x^i}\right)^2 + \langle \varrho_{su(N)}, \varrho_{su(N)}\rangle - \frac{1}{2}\sum_{1\leqslant i<j\leqslant N}\frac{\gamma(\gamma+1)}{\sin^2(x_i - x_j)}, \tag{5.29}$$

where $x := \frac{q}{2}$ with the real Cartan variable $q \in \mathcal{H}_{\text{r}}$ introduced before, and x^i denote the components of $x = \text{diag}(x_1, \ldots, x_N)$ in an orthonormal basis of $\mathcal{H}_{\text{r}}$ (here $\mathcal{T}_N = \mathrm{i}\mathcal{H}_{\text{r}} \subset su(N)$, see also footnote 6).

Recall that the standard N-particle Sutherland Hamiltonian, after separation of the center of mass and introducing dimensionless variables, can be written as follows:

$$H^{N,g}_{\text{standard}} = -\frac{1}{2}\sum_{i=1}^{N-1}\left(\frac{\partial}{\partial x^i}\right)^2 + \sum_{1\leqslant i<j\leqslant N}\frac{g(g-1)}{\sin^2(x_i - x_j)}, \tag{5.30}$$

where the domain of x is bounded by the singular locus of the potential. By setting $g := (\gamma+1)$, the comparison shows that

$$H^{N,g}_{\text{standard}} = -2\Delta^{\gamma}_{\text{red}} + 2\langle \varrho_{su(N)}, \varrho_{su(N)}\rangle. \tag{5.31}$$

It follows from (5.28) that

$$\text{spectrum}\big(\Delta^{\gamma}_{\text{red}}\big) = \{-\langle(\gamma\varrho_{su(N)} + \mu) + 2\varrho_{su(N)}, (\gamma\varrho_{su(N)} + \mu)\rangle | \mu \in L^+\}. \tag{5.32}$$

Combining the last two equations we recover the well-known formula (see, e.g., [8])

$$\text{spectrum}\big(H^{N,g}_{\text{standard}}\big) = \{2\langle \mu + g\varrho_{su(N)}, \mu + g\varrho_{su(N)}\rangle | \mu \in L^+\} \tag{5.33}$$

as a consequence of the representation theory [15]; for integer coupling constants $g > 0$.

5.2.2. The spectrum in the twisted $su(N)$ case with $\nu = k\lambda_1$. The non-trivial diagram automorphism θ satisfies

$$\Lambda^* = \Lambda \circ \theta \tag{5.34}$$

for any dominant integral weight of $su(N)$. This means that

$$\Lambda^* = \sum_{i=1}^{N-1} M^{N-i}\lambda_i \qquad \text{for} \qquad \Lambda = \sum_{i=1}^{N-1} M^i \lambda_i, \qquad M^i \in \mathbb{Z}_+. \tag{5.35}$$

Therefore, to find the spectrum (4.13) for the operators Δ_{red} in (5.19) and (5.20), we have to determine the admissible weights Λ for which

$$N^{\Lambda^*}_{\Lambda, k\lambda_1} = 1. \tag{5.36}$$

Theorem 5.3. *The dominant integral weights* Λ *of* $su(N)$ *that satisfy (5.36) have the form*

$$\Lambda = \mu + \chi, \tag{5.37}$$

where μ is an arbitrary self-conjugate dominant integral weight, $\mu^* = \mu$, and χ reads

$$\chi := \sum_{i=1}^{N-1} C^{i+1}\lambda_i \tag{5.38}$$

with some $(C^1, \ldots, C^N) \in \mathbb{Z}_+^N$ subject to the requirements

$$C^{N+1-i} = C^i \qquad \forall i = 1, \ldots, N \qquad \textit{and} \qquad \sum_{i=1}^{N} C^i = k. \tag{5.39}$$

The number of distinct solutions for χ equals $\dim\big(V_{k\lambda_1}^{\mathcal{T}_N^+}\big)$, which in particular means that k must be even if $N \geqslant 4$ is even. The spectrum of the corresponding twisted spin Sutherland Hamiltonian, given by (5.19) or (5.20), is provided by (4.13) with these weights Λ.

Proof. Taking any $\Lambda = \sum_{i=1}^{N-1} M^i\lambda_i$ satisfying (5.36), the Pieri formula (5.23) implies that

$$M^{N-i} = M^i + C^i - C^{i+1} \tag{5.40}$$

for some $C^i \in \mathbb{Z}_+$ subject to (5.24). If we define

$$\mu^i := M^i - C^{i+1}, \qquad i = 1, \ldots, N-1, \tag{5.41}$$

then we see from (5.24) that $\mu^i \in \mathbb{Z}_+$, and (5.40) gives

$$\mu^i = \mu^{N-i} + (C^{N+1-i} - C^i), \qquad \forall i = 1, \ldots, N-1. \tag{5.42}$$

Replacing i by $(N-i)$, we obtain

$$\mu^i = \mu^{N-i} + (C^{(N+1)-(i+1)} - C^{i+1}), \qquad \forall i = 1, \ldots, (N-1). \tag{5.43}$$

If we now define

$$\vartheta^i := C^{N+1-i} - C^i \qquad \forall i = 1, \ldots, N, \tag{5.44}$$

then we can read off from (5.42) and (5.43) that $\vartheta^i = \vartheta^{i+1}$, and hence

$$\vartheta^1 = \vartheta^2 = \ldots = \vartheta^N. \tag{5.45}$$

But by its very definition ϑ^i satisfies

$$\vartheta^{N+1-i} = C^i - C^{N+1-i} = -\vartheta^i, \tag{5.46}$$

and therefore $\vartheta^i = 0$, that is (5.39) holds. By substituting this back into equation (5.43), we obtain $\mu^i = \mu^{N-i}$. Thus (5.41) yields the required formula (5.37) with

$$\mu := \sum_{i=1}^{N-1} \mu^i\lambda_i, \qquad \chi := \sum_{i=1}^{N-1} C^{i+1}\lambda_i, \tag{5.47}$$

where $\mu^* = \mu$ and χ satisfies (5.39). We have just shown that all 'admissible weights' must have the form of Λ in (5.37), and it is trivial to check that any Λ of this form satisfies (5.36) indeed. The claim about the number of solutions for χ follows by obvious comparison of (5.39) with the bases of $\mathcal{V}_{k\lambda_1}^{\mathcal{T}_N^+}$ furnished by (5.12) and (5.15). □

6. Conclusion

In this paper, we first reviewed the quantum Hamiltonian symmetry reductions of the free particle on a Riemannian manifold governed by the Laplace–Beltrami operator. We assumed that the underlying symmetry group G is compact and it acts in a polar manner in the sense of Palais and Terng [13]. This strong assumption permits to derive nice results in a unified way and it holds in many important examples. For instance, it holds for the so-called Hermann actions (3.2), which are hyperpolar and lead to spin Sutherland-type models in general.

In the main text we focused on the group action defined by twisted conjugations (3.12) and described the corresponding twisted spin Sutherland models in proposition 3.1. Then we explained how to diagonalize these reduced Hamiltonians in principle and have characterized their spectra explicitly in theorem 5.3 for certain special cases associated with $G = SU(N)$. These special cases were obtained by using the symmetric tensorial powers of the defining representation of $SU(N)$ in the definition of the reduction, and we also presented 'oscillator realizations' of the resulting Hamiltonians in propositions 5.1 and 5.2.

The classical analogs of the twisted spin Sutherland models have been investigated in [20]. Further related studies of classical Hamiltonian reduction can be found in [19, 21] and references therein. A comparison between the outcomes of the classical and quantum Hamiltonian reductions under polar actions in general was given in [11].

The reduced systems that feature in proposition 2.1 possess a hidden generalized Weyl group symmetry, as was pointed out in remark 2.3. The structure of the generalized Weyl group belonging to the twisted conjugations (3.12) has been clarified in [23, 24], and we outline this in appendix B giving an explicit description in the $SU(N)$ cases. The other two appendices contain some technical details relegated from the main text.

There remain several open problems requiring future work. First, it would be interesting to find the eigenfunctions for the models studied in section 5. We note that generalized spin Sutherland models with similar interaction potentials, but different spin variables, have been constructed by Finkel *et al* [36, 37] by means of the differential-difference operator technique. It would be desirable to better understand the relationship between our models and those in [36, 37] (see also [38]). A relevant fact in this respect is that if θ is an involution of a classical Lie algebra, then the 'twisted Weyl group' (3.31) contains a standard Weyl group of B-type as a subgroup (see appendix B). It appears worthwhile to enquire about the use of the twisted Weyl groups in constructing Dunkl-type operators. Second, it is a natural task to describe the full family of spin Sutherland models, and in particular all spinless special cases, that can be associated with the Hermann actions (3.2). These also include non-compact Riemannian manifolds as starting points, and the scattering theory of the associated many-body models with spin appears worth investigating. Third, the assumptions made in section 2 can be weakened and modified in various ways. For example, one should extend the formalism to pseudo-Riemannian manifolds, such as semisimple group manifolds of non-compact type (see [18]). The strong conditions that we required in the definition of a 'section' for convenience should be relaxed, as is done in some works in the theory of polar actions, since there exist examples for which these assumptions are not valid. Finally, it is a challenge to understand whether an infinite-dimensional analog of the formalism of polar actions can play a role in describing Calogero-type models with elliptic interaction potentials.

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Appendix A. On the 'twisted measure factor' in equation (3.28)

In this appendix, we explain that the second term (the so-called measure factor) appearing in formula (2.17) of the reduced Laplace–Beltrami operator gives just a *constant* for the twisted spin Sutherland models as stated by equation (3.28). In the non-twisted cases the analogous result is well known [16, 14], and it turns out that the 'twisted measure factors' can be traced back to the non-twisted ones. The density function associated with twisted conjugations was also calculated in [22], and our arguments below follow from those in this reference.

Keeping the notations introduced in section 3, consider the real Cartan subalgebra $\mathcal{H}_{\mathrm{r}}$ of a complex simple Lie algebra $\mathcal{A}$ with scalar product $\langle , \rangle$ and associated root system Φ. Denote by $\Delta_{\mathcal{H}_{\mathrm{r}}}$ the Laplace–Beltrami operator of $(\mathcal{H}_{\mathrm{r}}, \langle , \rangle)$, using the restricted scalar product[8]. Define $\varrho_{\mathcal{G}} := \frac{1}{2}\sum_{\alpha\in\Phi_+}\alpha$, where $\mathcal{G}$ is the compact real form and $\varrho_{\mathcal{G}} = \varrho_{\mathcal{G}}^{\mathrm{id}}$ in the notation of (3.29). Introduce the standard 'density function' $\delta_{\Phi_+}^{\frac{1}{2}} : \mathcal{H}_{\mathrm{r}} \to \mathbb{R}$ by

$$\delta_{\Phi_+}^{\frac{1}{2}}(q) := \prod_{\alpha\in\Phi_+} \sin\left(\frac{\alpha(q)}{2}\right). \tag{A.1}$$

It is well known [16, 14] that the following identity holds:

$$\Delta_{\mathcal{H}_{\mathrm{r}}}\delta_{\Phi_+}^{\frac{1}{2}} = -\langle \varrho_{\mathcal{G}}, \varrho_{\mathcal{G}}\rangle \delta_{\Phi_+}^{\frac{1}{2}}. \tag{A.2}$$

This identity is independent of the normalization of $\langle , \rangle$, and the value of the constant can be found from Freudenthal's strange formula

$$g_{\mathcal{G}} \dim(\mathcal{G}) = 12\langle \varrho_{\mathcal{G}}, \varrho_{\mathcal{G}}\rangle_0, \tag{A.3}$$

where $g_{\mathcal{G}}$ is the dual Coxeter number of $\mathcal{G}$ and $\langle , \rangle_0$ is the scalar product normalized so that the length of the long roots is $\sqrt{2}$. Since the expressions in (3.25) and (A.1) are equal if $\theta = \mathrm{id}$, (A.2) yields the measure factor for the non-twisted spin Sutherland models.

To handle the twisted cases based on the non-trivial involutive diagram automorphisms of the classical Lie algebras, one may show by case-by-case inspection that the 'twisted density function' (3.25) satisfies

$$\delta^{\frac{1}{2}}(\mathrm{e}^{\mathrm{i}q}) = \prod_{\alpha\in\mathfrak{R}_+} \sin\left(\frac{\alpha(q)}{2}\right) \prod_{\lambda\in\mathfrak{W}_+} \cos\left(\frac{\lambda(q)}{2}\right) = \mathcal{C}\prod_{\varphi\in P_+} \sin\left(\frac{\varphi(q)}{2}\right), \tag{A.4}$$

where $\mathcal{C}$ is a constant and the set P_+ is given in terms of a corresponding root systems as follows:

$$P_+ = \begin{cases} \Phi_+(C_n) & \text{if} \quad \mathcal{A} = D_{n+1}, \\ 2\Phi_+(B_n) & \text{if} \quad \mathcal{A} = A_{2n-1}, \\ 2\Phi_+(C_n) & \text{if} \quad \mathcal{A} = A_{2n}. \end{cases} \tag{A.5}$$

This formula, which is equivalent to the formula of the density function given in [22], is obtained from (3.25) by using that $\sin(2x) = 2\sin(x)\cos(x)$. Referring to (3.29) and P_+ in (A.5), it is also readily verified that

$$\varrho_{\mathcal{G}}^{\theta} = \frac{1}{2}\sum_{\alpha\in\mathfrak{R}_+}\alpha + \frac{1}{2}\sum_{\lambda\in\mathfrak{W}_+}\lambda = \frac{1}{2}\sum_{\varphi\in P_+}\varphi. \tag{A.6}$$

[8] In coordinates q^i of $q \in \mathcal{H}_{\mathrm{r}}$ with respect to an orthonormal basis of $\mathcal{H}_{\mathrm{r}}$, $\Delta_{\mathcal{H}_{\mathrm{r}}} = \sum_i \partial_i^2$, similarly to (3.27).

Equations (A.4) and (A.5) show that the twisted density function can be cast into a non-twisted form. Since the non-twisted measure factors are constants, one may conclude that the twisted measure factors are constants too. As P_+ appears both in (A.4) and (A.6), one obtains the identity (3.28) from (A.2) applied to $\left(\mathcal{A}^+, \mathcal{H}_r^+\right)$. With the scalar product (5.2), in the twisted $\mathcal{G} = su(N)$ cases the values of the constants on the right-hand side of (3.28) are found from the relations

$$6\langle \varrho_{\mathcal{G}}^{\theta}, \varrho_{\mathcal{G}}^{\theta}\rangle = \begin{cases} n(2n-1)(2n+1) & \text{if} \quad \mathcal{G} = su(2n), \\ 2n(n+1)(2n+1) & \text{if} \quad \mathcal{G} = su(2n+1). \end{cases} \tag{A.7}$$

These relations can be checked by explicit calculation or by application of the strange formula (A.3) taking care of the normalizations involved.

The measure factors are constants for all diagram automorphisms of E_6 and D_4 as well. In fact, the non-twisted form of the density function is established in [22] in these cases, too.

Appendix B. Generalized Weyl groups for twisted conjugations

For completeness, we here recall from [22–24] the structure of the generalized Weyl group associated with the twisted conjugation action (3.12). As in the main text, we assume that G is simply connected and Θ is involutive.

In this case (2.18) becomes the 'twisted Weyl group' $W(G, T^{\Theta}, \Theta)$ in (3.31). Let $N_G(T)$ denote the normalizer of T in G defined in the usual manner, i.e. $N_G(T^{\Theta}, I^{\Theta})$ in (3.32) yields $N_G(T)$ for $\Theta = \mathrm{id}$, and also consider $N_{G^{\Theta}}(T^{\Theta})$, where G^{Θ} is the (connected) fixed point subgroup of Θ in G. According to [23], the twisted Weyl group is given by the semidirect product

$$W(G, T^{\Theta}, \Theta) \cong W(G^{\Theta}, T^{\Theta}) \ltimes (T/T^{\Theta})^{\Theta}. \tag{B.1}$$

Here $W(G^{\Theta}, T^{\Theta}) = N_{G^{\Theta}}(T^{\Theta})/T^{\Theta}$ and the finite Abelian group $(T/T^{\Theta})^{\Theta}$ is formed by the fixed points of the induced action of Θ on (T/T^{Θ}). Note that $W(G^{\Theta}, T^{\Theta})$ acts by automorphisms of $(T/T^{\Theta})^{\Theta}$ naturally as

$$(gT^{\Theta}) \cdot (tT^{\Theta}) = gtg^{-1}T^{\Theta} \quad \text{for any} \quad gT^{\Theta} \in W(G^{\Theta}, T^{\Theta}), \qquad tT^{\Theta} \in (T/T^{\Theta})^{\Theta}. \tag{B.2}$$

This is well defined since $N_{G^{\Theta}}(T^{\Theta}) < N_G(T)$, which follows from the fact [31] that T^{Θ} contains a regular element of G (whose centralizer is T). As induced from the I^{Θ} action (3.12), the subgroup $W(G^{\Theta}, T^{\Theta})$ in (B.1) acts on T^{Θ} by its ordinary Weyl group action. The map $S : (T/T^{\Theta})^{\Theta} \to T^{\Theta}$ defined by

$$S : tT^{\Theta} \mapsto S(tT^{\Theta}) := \Theta(t)t^{-1} \tag{B.3}$$

is an injective homomorphism, and the normal subgroup $(T/T^{\Theta})^{\Theta}$ in (B.1) acts on the 'section' T^{Θ} by the translations generated by its image in T^{Θ}:

$$(tT^{\Theta}) \cdot t_0 = S(tT^{\Theta})t_0 \quad \text{for any} \quad tT^{\Theta} \in (T/T^{\Theta})^{\Theta}, \qquad t_0 \in T^{\Theta}. \tag{B.4}$$

Correspondingly, (B.1) can be rewritten in the form

$$W(G, T^{\Theta}, \Theta) \cong W(G^{\Theta}, T^{\Theta}) \ltimes S((T/T^{\Theta})^{\Theta}), \tag{B.5}$$

where $W(G^{\Theta}, T^{\Theta})$ acts naturally on the finite subgroup $S((T/T^{\Theta})^{\Theta}) < T^{\Theta}$.

To proceed further, let us introduce the lattices

$$\mathcal{L} := \mathrm{Ker}\,(\exp|_{\mathcal{T}}) := \{X \in \mathcal{T} \,|\, \exp(X) = e\} < \mathcal{T} \qquad \text{and} \qquad \mathcal{L}^{\theta} := \mathcal{L} \cap \mathcal{T}^{\theta}, \tag{B.6}$$

where now $\mathcal{T}^{\theta}$ denotes the fixed point set of θ in $\mathcal{T}$. Note that $T \cong \mathcal{T}/\mathcal{L}$ and $T^{\Theta} \cong \mathcal{T}^{\theta}/\mathcal{L}^{\theta}$. Let $p : \mathcal{T} \to \mathcal{T}^{\theta}$ be the orthogonal projection with respect to the Killing form. Then [24] one has yet another natural isomorphism:

$$S((T/T^{\Theta})^{\Theta}) \cong p(\mathcal{L})/\mathcal{L}^{\theta}. \tag{B.7}$$

To understand this, first note that if $\theta(X_-) = -X_-$, then $\Theta\big(\exp\left(\frac{1}{2}X_-\right)T^\Theta\big) = \exp\left(\frac{1}{2}X_-\right)T^\Theta$ is equivalent to $\exp(-X_-) \in T^\Theta$, which requires that $\exp(-X_-) = \exp(X_+)$ for some $X_+ \in \mathcal{T}^\theta$. Next, if $X = (X_+ + X_-) \in \mathcal{L}$ with $X_+ = p(X) \in \mathcal{L}^\theta$, then $\exp\left(\frac{1}{2}X_-\right)T^\Theta \in (T/T^\Theta)^\Theta$ and

$$S\left(\exp\left(\tfrac{1}{2}X_-\right)T^\Theta\right) = \exp(-X_-) = \exp(X_+) \in p(\mathcal{L})/\mathcal{L}^\theta < T^\Theta \cong \mathcal{T}^\theta/\mathcal{L}^\theta. \tag{B.8}$$

Every element of $p(\mathcal{L})/\mathcal{L}^\theta$ corresponds by (B.8) to a unique element of $S((T/T^\Theta)^\Theta)$. Taking into account (B.7), the semidirect product

$$W(G, T^\Theta, \Theta) \cong W(G^\Theta, T^\Theta) \ltimes p(\mathcal{L})/\mathcal{L}^\theta \tag{B.9}$$

can be 'lifted' to yield the infinite group

$$W(G^\Theta, T^\Theta) \ltimes p(\mathcal{L}), \tag{B.10}$$

which can be identified [22, 24] as the Weyl group of the twisted affine Lie algebra associated with the pair $(\mathcal{A}, \theta)$. The twisted conjugacy classes in G are parametrized by the orbits of the twisted Weyl group in the section T^Θ, i.e., they form the (stratified) space

$$[\mathcal{T}^\theta/\mathcal{L}^\theta]/[W(G^\Theta, T^\Theta) \ltimes p(\mathcal{L})/\mathcal{L}^\theta] \cong \mathcal{T}^\theta/[W(G^\Theta, T^\Theta) \ltimes p(\mathcal{L})]. \tag{B.11}$$

The set on the right-hand side of (B.11) is the space of orbits for the twisted affine Weyl group acting on $\mathcal{T}^\theta$, which is described in [31]. By using this, Mohrdieck and Wendt [24] gave parametrizations for the twisted conjugacy classes of any simply connected, connected G. In our cases of interest, this is equivalently given by the domains listed at the end of section 3.

Let us display the twisted Weyl groups in the picture (B.5) for $G = SU(N)$. The fixed point sets T_N^Θ in the maximal torus T_N of $SU(N)$ are

$$\begin{aligned} T_{2n}^\Theta &= \big\{t \,|\, t = \mathrm{diag}\big(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}\big), |t_i| = 1, 1 \leqslant i \leqslant n\big\}, \\ T_{2n+1}^\Theta &= \big\{t \,|\, t = \mathrm{diag}\big(t_1, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1}\big), |t_i| = 1, 1 \leqslant i \leqslant n\big\}. \end{aligned} \tag{B.12}$$

In both cases

$$W\big(SU(N)^\Theta, T_N^\Theta\big) \cong W_{B_n} \cong S_n \ltimes (\mathbb{Z}_2)^{\times n}. \tag{B.13}$$

The symmetric group S_n permutes the components $t_1, \ldots, t_n$ of $t \in T_N^\Theta$ and, for any fixed $1 \leqslant i \leqslant n$, the generator $\tau_i \in (\mathbb{Z}_2)^{\times n}$ exchanges the components t_i and t_i^{-1} of t. It is easy to compute that $S\big(\big(T_N/T_N^\Theta\big)^\Theta\big)$ consists of the elements σ of the form

$$\begin{aligned} &\sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n, \sigma_n, \ldots, \sigma_1) \quad \text{with} \quad \sigma_i \in \{\pm 1\}, \quad \prod_{i=1}^n \sigma_i = 1 \quad \text{if} \quad N = 2n, \\ &\sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n, 1, \sigma_n, \ldots, \sigma_1) \quad \text{with} \quad \sigma_i \in \{\pm 1\} \quad \text{if} \quad N = (2n+1). \end{aligned} \tag{B.14}$$

Thus we see that

$$\begin{aligned} W\big(SU(2n)^\Theta, T_{2n}^\Theta, \Theta\big) &\cong W_{B_n} \ltimes (\mathbb{Z}_2)^{\times(n-1)}, \\ W\big(SU(2n+1)^\Theta, T_{2n+1}^\Theta, \Theta\big) &\cong W_{B_n} \ltimes (\mathbb{Z}_2)^{\times n}. \end{aligned} \tag{B.15}$$

The S_n subgroup of W_{B_n} acts by permuting the components σ_i of $\sigma \in S\big(\big(T_N/T_N^\Theta\big)^\Theta\big)$, while the generators τ_i act trivially on $S\big(\big(T_N/T_N^\Theta\big)^\Theta\big)$.

Finally, let us remark that one may introduce (physically and mathematically different) 'extended versions' of the twisted spin Sutherland models of proposition 3.1 by postulating the dense open subset $(\exp|_\mathcal{T})^{-1}\,(\hat{T}^\Theta) \subset \mathcal{T}^\theta$ to be the configuration space, instead of the alcove $\check{\mathcal{T}}^\theta$. For suitable domains of the formal Hamiltonian obtained from (3.30), these extended models possess the twisted affine Weyl group (B.10) as a symmetry group.

Appendix C. On the derivation of propositions 5.1 and 5.2

We here sketch how we obtained the explicit formulae (5.19) and (5.20) for the oscillator realizations of the reduced Hamiltonian operators in the twisted $SU(N)$ cases. By applying the conventions of section 3, we first note that both for $\varphi \in \mathfrak{R}_+$ and $\varphi \in \mathfrak{W}_+$ we have

$$\rho'\big(Y^{\pm}_{\varphi}\big)^2 + \rho'\big(Z^{\pm}_{\varphi}\big)^2 = -\rho'\big(X^{\pm}_{\varphi}\big)\rho'\big(X^{\pm}_{-\varphi}\big) - \rho'\big(X^{\pm}_{-\varphi}\big)\rho'\big(X^{\pm}_{\varphi}\big). \tag{C.1}$$

To spell out Δ_{red} (3.30) in terms of the oscillators in (5.6), we need matrix realizations for the generators $X^{\pm}_{\varphi}$ and K^{-}_{j}. The involution θ (5.11) preserves the Cartan subalgebra $\mathcal{H}$ spanned by diagonal matrices, while the root and weight vectors in (C.1) are off-diagonal matrices. It is enough to list the latter base elements for positive φ, since $X^{\pm}_{-\varphi} = \big(X^{\pm}_{\varphi}\big)^T$ holds.

When N is even, $N = 2n$, $\mathcal{H}^{+}_{\text{r}}$ consists of the elements (3.36) and in $\mathcal{H}^{-}_{\text{r}}$ we define an orthonormal basis $\{K^{-}_{j}\}_{j=1}^{n-1}$ by the matrices

$$K^{-}_{j} := \frac{1}{\sqrt{2j(j+1)}}\left(\sum_{i=1}^{j}(E_{i,i} + E_{2n+1-i,2n+1-i}) - j(E_{j+1,j+1} + E_{2n-j,2n-j})\right). \tag{C.2}$$

As one can easily verify, the following off-diagonal matrices,

$$\begin{aligned}
X^{+}_{2e_k} &:= E_{k,2n+1-k} \qquad (1 \leqslant k \leqslant n),\\
X^{\pm}_{e_k-e_l} &:= \tfrac{1}{\sqrt{2}}(E_{k,l} \mp (-1)^{k+l}E_{2n+1-l,2n+1-k}) \qquad (1 \leqslant k < l \leqslant n),\\
X^{\pm}_{e_k+e_l} &:= \tfrac{1}{\sqrt{2}}(E_{k,2n+1-l} \pm (-1)^{k+l}E_{l,2n+1-k}) \qquad (1 \leqslant k < l \leqslant n)
\end{aligned} \tag{C.3}$$

provide root and weight vectors satisfying the conventions of subsection 3.1, using the scalar product (5.2).

When N is odd, $N = 2n+1$, $\mathcal{H}^{+}_{\text{r}}$ is parametrized according to (3.39) and in $\mathcal{H}^{-}_{\text{r}}$ we introduce dual bases $\{L_j\}_{j=1}^{n}, \{L^j\}_{j=1}^{n}$ defined by

$$L_j := E_{j,j} + E_{2n+2-j,2n+2-j} - 2E_{n+1,n+1}, \qquad L^j := \frac{1}{2}L_j - \frac{1}{2n+1}\sum_{i=1}^{n}L_i. \tag{C.4}$$

These permit to evaluate the third term of (3.30) by means of the relation

$$\sum_j \rho'(\mathrm{i}K^{-}_{j})^2 = -\sum_j \rho'(L^j)\rho'(L_j). \tag{C.5}$$

The normalized root and weight vectors in $sl(2n+1,\mathbb{C})$ are furnished by the matrices

$$\begin{aligned}
X^{-}_{2e_k} &:= E_{k,2n+2-k} \qquad (1 \leqslant k \leqslant n),\\
X^{\pm}_{e_k} &:= \tfrac{1}{\sqrt{2}}(E_{k,n+1} \pm (-1)^{k+n}E_{n+1,2n+2-k}) \qquad (1 \leqslant k \leqslant n),\\
X^{\pm}_{e_k-e_l} &:= \tfrac{1}{\sqrt{2}}(E_{k,l} \mp (-1)^{k+l}E_{2n+2-l,2n+2-k}) \qquad (1 \leqslant k < l \leqslant n),\\
X^{\pm}_{e_k+e_l} &:= \tfrac{1}{\sqrt{2}}(E_{k,2n+2-l} \mp (-1)^{k+l}E_{l,2n+2-k}) \qquad (1 \leqslant k < l \leqslant n).
\end{aligned} \tag{C.6}$$

Based on the above introduced generators and equation (5.6), the explicit formulae displayed in propositions 5.1 and 5.2 follow by straightforward calculation.

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